Reduced filtered K-theory for Cuntz-Krieger algebras

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Cuntz-Krieger algebras

Definition (Cuntz-Krieger)

Let A be a non-degenerate $n \times n$ matrix over $\{0,1\}$. Its associated *Cuntz-Krieger algebra* $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $s_1, \ldots, s_n$ satisfying the relations:

- $1_{\mathcal{O}_A} = s_1 s_1^* + \cdots + s_n s_n^*$
- $s_i^* s_i = \sum_{j=1}^n A(i,j) s_j s_j^*$ for all $i \in \{1, \ldots, n\}$

From the matrix A one can compute:

- the primitive ideal space $\mathrm{Prim}(\mathcal{O}_A)$ of $\mathcal{O}_A$,
- the K-theory of $\mathcal{O}_A$.

The following are equivalent (and can be determined from A):

- $\mathcal{O}_A$ is $\mathcal{O}_\infty$-absorbing,
- $\mathcal{O}_A$ has real rank zero,
- $\mathrm{Prim}(\mathcal{O}_A)$ is finite.

Reduced filtered K-theory

A C^*-algebra $\mathfrak{A}$ can be given a structure as a *C^*-algebra over a topological space X* via a continuous map $\mathrm{Prim}(\mathfrak{A}) \to X$.

- A C^*-algebra $\mathfrak{A}$ is canonically a C^*-algebra over its primitive ideal space $\mathrm{Prim}(\mathfrak{A})$.
- We write $Y \mapsto \mathfrak{A}(Y)$ for the induced map from openlocally closed subsets of X to idealssubquotients in $\mathfrak{A}$.
- For X finite, U_x denotes the smallest open neighborhood of x in X.
- We write $x \to y$ for $x, y \in X$ when $\overline{\{x\}} \supsetneq \overline{\{y\}}$ and there is no $z \in X$ for which $\overline{\{x\}} \supsetneq \overline{\{z\}} \supsetneq \overline{\{y\}}$.

Definition (Boyle-Huang, Restorff)

Let $\mathfrak{A}$ be a C^*-algebra over a finite T_0-space X. Its *reduced filtered K-theory $FK_{\mathcal{R}}(\mathfrak{A})$* consists of

$$K_1(\mathfrak{A}(x)) \xrightarrow{\delta} K_0(\mathfrak{A}(U_x \setminus \{x\}) \xrightarrow{i} K_0(\mathfrak{A}(U_x))$$

for all $x \in X$, plus the maps

$$K_0(\mathfrak{A}(U_x)) \xrightarrow{i} K_0(A(U_y \setminus \{y\})$$

for all $x, y \in X$ with $x \to y$.

Classification of Cuntz-Krieger algebras

Theorem (Rørdam, Restorff)

Let $\mathcal{O}_A$ and $\mathcal{O}_B$ be Cuntz-Krieger algebras with finite primitive ideal space X. Then $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_B \otimes \mathbb{K}$ if and only if $FK_{\mathcal{R}}(\mathcal{O}_A) \cong FK_{\mathcal{R}}(\mathcal{O}_B)$.

Theorem (Eilers-Katsura-Tomforde-West, A-Bentmann-Katsura)

Let X be a finite T_0-space and $\mathfrak{B}$ a C^*-algebra over X of real rank zero. Assume for all $x \in X$ that

- *$K_1(\mathfrak{B}(x))$ is free,*
- *$K_*(\mathfrak{B}(x))$ is finitely generated,*
- $\operatorname{rank} K_1(\mathfrak{B}(x)) = \operatorname{rank} K_0(\mathfrak{B}(x))$.

Then there exists a Cuntz-Krieger algebra $\mathcal{O}_A$ with $\operatorname{Prim}(\mathcal{O}_A) \cong X$ and $FK_{\mathcal{R}}(\mathcal{O}_A) \cong FK_{\mathcal{R}}(\mathfrak{B})$.

Strong and external classification

A separable, nuclear, $\mathcal{O}_\infty$-absorbing C^*-algebra $\mathfrak{A}$ with $\mathrm{Prim}(\mathfrak{A}) \cong X$ is called a *Kirchberg X-algebra*.

Theorem (Kirchberg, Meyer-Nest, Bentmann-Köhler, A-Restorff-Ruiz, A-Bentmann-Katsura)

Let X be a connected T_0-space and assume that either X is accordion or $|X| \leq 4$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kirchberg X-algebras of real rank zero. Assume for all $x \in X$ that $K_1(\mathfrak{A}(x))$ and $K_1(\mathfrak{B}(x))$ are free, and that $\mathfrak{A}(x)$ and $\mathfrak{B}(x)$ are in the bootstrap category. Then any isomorphism $FK_{\mathcal{R}}(\mathfrak{A}) \to FK_{\mathcal{R}}(\mathfrak{B})$ can be lifted to an X-equivariant isomorphism $\mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$.

Theorem (A-Restorff-Ruiz)

There exists a Cuntz-Krieger algebra $\mathcal{O}_A$ with $|\mathrm{Prim}(\mathcal{O}_A)| = 4$ for which the canonical map

$$\mathrm{Aut}(\mathcal{O}_A \otimes \mathbb{K}) \to \mathrm{Aut}(FK(\mathcal{O}_A))$$

is not surjective.