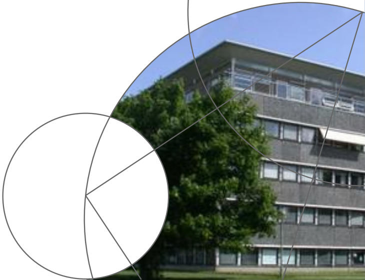
Faculty of Science

Classification of nonsimple C^*-algebras of real rank zero

Sara Arklint
Department of Mathematical Sciences



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The most desirable questions to answer are usually:

- Is F a complete invariant? I.e., does $F(A) \cong F(B)$ imply $A \cong B$?
- What is the range of F? I.e., exactly which objects in $\mathfrak{C}$ occur as $F(A)$ for some object A in $\mathfrak{Ob}$?

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Objects

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Any stabilized Cuntz-Krieger algebra satisfying property (II) is contained in $\mathfrak{K}\mathfrak{A}(X)$ for suitable X. Any stabilized, purely infinite graph algebra with finitely many ideals is contained in $\mathfrak{K}\mathfrak{A}(X)$ for suitable X.

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Invariant

For a C^*-algebra A in $\mathfrak{K}\mathfrak{A}(X)$, the filtered K-theory $\mathrm{FK}(A)$ intuitively consists of all groups and maps appearing in

$$\begin{array}{ccccc} K_0(J/I) & \longrightarrow & K_0(K/I) & \longrightarrow & K_0(K/J) \\ \uparrow\circ & & & & \circ\downarrow \\ K_1(K/J) & \longleftarrow & K_1(K/I) & \longleftarrow & K_1(J/I) \end{array}$$

arising from $J/I \hookrightarrow K/I \twoheadrightarrow K/J$ for all $I \trianglelefteq J \trianglelefteq K \trianglelefteq A$.

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Theorem (Restorff)

Let A and B be stabilized Cuntz-Krieger algebras in $\mathfrak{KA}(X)$. Then $\mathsf{FK}(A) \cong \mathsf{FK}(B)$ implies $A \cong B$.

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Theorem (Bentmann-Köhler)

The following are equivalent for a finite, connected T_0-space X:

- *X is an accordion space.*
- *For all A and B in $\mathfrak{KA}(X)$, $\mathsf{FK}(A) \cong \mathsf{FK}(B)$ implies $A \cong B$.*

The proof relies heavily on work of Kirchberg and Meyer-Nest.

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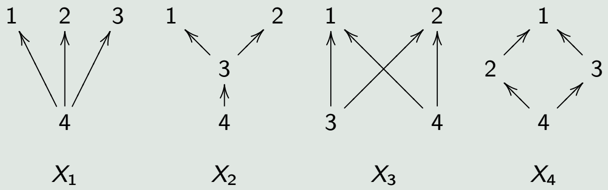
The smallest examples of nonaccordion spaces



On completeness of filtered K-theory

Theorem (A-Restorff-Ruiz)

For A and B in $\mathfrak{KA}(X_1)$ of real rank zero, $\mathsf{FK}(A) \cong \mathsf{FK}(B)$ *implies* $A \cong B$.

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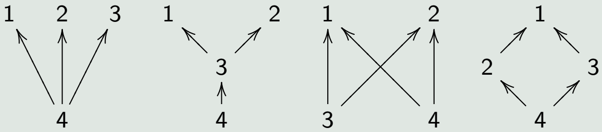

X_1 X_2 X_3 X_4

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For A and B in $\mathfrak{K}\mathfrak{A}(X_1)$ of real rank zero, $\mathsf{FK}(A) \cong \mathsf{FK}(B)$ *implies* $A \cong B$*.*

Theorem (Brown-Pedersen)

Consider an extension $I \hookrightarrow A \twoheadrightarrow A/I$ of C^*-algebras. Then A is of real rank zero if and only if I and A/I are of real rank zero and projections lift from A/I to A.

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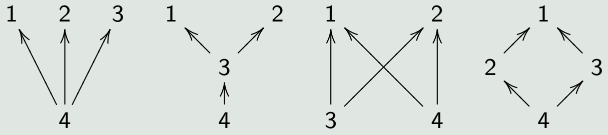

X_1 X_2 X_3 X_4

On completeness of reduced filtered K-theory

Theorem (A-Bentmann-Katsura)

Let A and B in $\mathfrak{KA}(X_4)$ of real rank zero be given. If $K_1(A(x))$ is free for all $x \in X_4$, then $\mathsf{FK}(A) \cong \mathsf{FK}(B)$ implies $A \cong B$.

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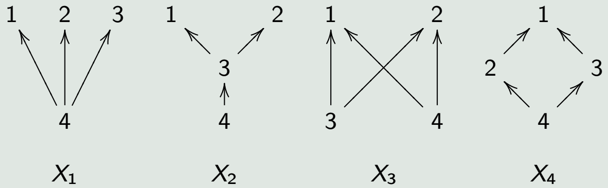

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Let A and B in $\mathfrak{KA}(X_4)$ of real rank zero be given. If $K_1(A(x))$ is free for all $x \in X_4$, then $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(B)$ implies $A \cong B$.

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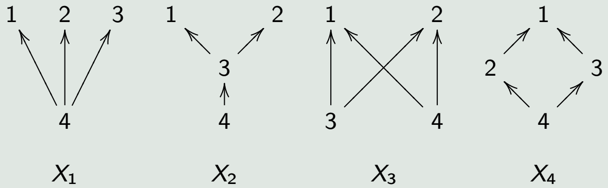

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Theorem (A-Bentmann-Katsura)

Let X be accordion or one of the spaces X_1 and X_2. If A and B in $\mathfrak{K}\mathfrak{A}(X)$ are of real rank zero and $K_1(A(x))$ is free for all $x \in X$, then $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(B)$ implies $A \cong B$.

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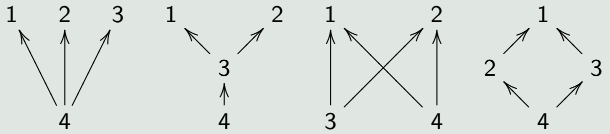

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On the range of filtered K-theory

Definition (Graph algebra)

For a countable, directed graph $E = (E^0, E^1, r, s)$, we let the $C^*(E)$ denote the universal C^*-algebra generated by the relations

$$\begin{aligned} p_v &= p_v^* = p_v^2 \\ p_v p_w &= 0 \text{ when } v \neq w \\ s_e^* s_e &= p_{s(e)} \\ p_v &= \sum_{e \in r^{-1}(v)} s_e s_e^* \text{ when } 0 < |r^{-1}(v)| < \infty \end{aligned}$$

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Theorem (A-Bentmann-Katsura)

Let A in $\mathfrak{KA}(X)$ be given and assume that $K_1(A(x))$ is free for all $x \in X$. Then there exists a countable, directed graph E for which

- $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(C^*(E))$.
- *$C^*(E)$ belongs to $\mathfrak{KA}(X)$.*

On extensions of stabilized Cuntz-Krieger algebras

Corollary

Let X be a suitably nice, finite T_0-space, and let A be a tight C^*-algebra over X. Consider an extension $I \hookrightarrow A \twoheadrightarrow A/I$. Then the following are equivalent:

- *A is a stabilized Cuntz-Krieger algebra.*
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- By the range result, $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(C^*(E))$ for $C^*(E)$ a Cuntz-Krieger algebra in $\mathfrak{KA}(X)$.
- By completeness of $\mathrm{FK}_{\mathcal{R}}$, $A \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. □