Thesis for the Master degree in Mathematics
Study board for Mathematical Sciences
Institute for Mathematical Sciences
University of Copenhagen

A tribute to $\mathrm{K}_0(-;\mathbb{Z}/n)$

Sara Arklint

Advisor: Søren Eilers
Submitted: December 14, 2007
Corrected: October 8, 2008

Dansk resumé

Nærværende speciale er en gennemgang af en K-teoretisk klassificering af en klasse af C^*-algebraer, de såkaldte AD algebraer. AD algebraerne er netop de C^*-algebraer som fås som tællelige direkte limites af endelige direkte summer af matrixalgebraer over cirkelalgebraen og dimensionfaldsalgebraer. Det vises ved et eksempel at klassisk K-teori bestående af de ordnede K-grupper *ikke* er en fuldstændig invariant for klassen af reel rang nul AD algebraer. Hovedresultatet i specialet er Søren Eilers' bevis for at en udvidet invariant, nærmere bestemt klassisk K-teori sammen med den ordnede K_0-gruppe med koefficienter i $\mathbb{Z}/n$ og de to naturlige transformationer, er en stærkt fuldstændig invariant for klassen af reel rang nul AD algebraer hvor torsionen i K_1-gruppen annihileres af n. Et billedresultat for invarianten inddrages og bruges til at vise at invarianten kan reduceres til klassisk K-teori når man restringerer til simple C^*-algebraer.

Abstract in English

In this thesis we give a thorough description of a K-theoretical classification of a class of C^*-algebras called the AD algebras. The class of AD algebras consists of the countable inductive limits of finite direct sums of matrix algebras over the circle algebra and the dimension drop algebras. By an example it is shown that classical K-theory, consisting of the ordered K-groups, is *not* a complete invariant for the class of real rank zero AD algebras. The main result is the proof by Søren Eilers that the invariant consisting of classical K-theory with the ordered K_0-group with coefficients in $\mathbb{Z}/n$ and its associated natural transformations augmented, is a strongly complete invariant for the class of real rank zero AD algebras for which n annihilates the torsion in the K_1-group. By a recently established range result for the invariant, we prove that when restricting to simple C^*-algebras one may reduce the invariant to classical K-theory.

Contents

0 Prologue

The main purpose of this thesis has been to give a thorough description of a solution to a small part of the wide problem of classifying C^*-algebras by K-theoretical invariants, namely the solution presented in [Eil95].

0.1 Concerning classification

A common approach to the problem is to consider but a small class of C^*-algebras and examine whether a given K-theoretical invariant suffices. As part of the examination of an invariant F, one may determine the *range* of the invariant, i.e. determine which objects $F(A)$ that appear when A runs through the whole class, and one may determine whether the invariant can be *reduced*, i.e. whether the invariant is still complete when one omits certain parts of it.

An invariant F is called *(weakly) complete* if $F(A) \cong F(B)$ implies $A \cong B$ for all C^*-algebras A and B in the given class. If furthermore any isomorphism $\varphi\colon F(A) \to F(B)$ can be lifted to an isomorphism $\alpha\colon A \to B$ (such that $F(\alpha) = \varphi$), then the invariant is called *strongly complete*.

The perhaps most strong and simply formulated classification result was proven by E. Kirchberg and N. C. Phillips independently around 1994. It states that the group $\mathrm{K}_*(-)$ is a strongly complete invariant for the class of purely infinite, separable, nuclear C^*-algebras for which the UCT holds. Most classification results, however, concern only certain *inductive limit classes*, i.e. classes of countable inductive limits of small collections of C^*-algebras.

A classical result proven by G. Elliott in 1976, [Ell76], is that the invariant consisting of the group $\mathrm{K}_0(-)$ together with its positive cone $\mathrm{K}_0(-)^+$ and the scale $\Sigma(-) \subseteq \mathrm{K}_0(-)^+$ is a strongly complete invariant for the class of AF algebras.

The class of AF algebras consists of the countable inductive limits of finite direct sums of matrix algebras. And during the last thirty years, the agenda has been to expand to larger inductive limit classes and see whether the invariant sufficies, and how to expand it when is does not suffice.

Now, the AF algebras, as well as the simple purely infite C^*-algebras, have real rank zero. A condition one usually has to demand of one's C^*-algebras to achieve a K-theoretical classification result. Real rank zero guaranties the existence of many projections and therefore lots of information in the K_0-group. However, in the case, for example, of a simple inductive limit of homogeneous C^*-algebras, real rank zero will force the connecting maps of the system to approximately act as the maps in a system of finite direct sums of matrix algebras. This illustrates how harsh a demand real rank zero is, and it somewhat explains why one is more capable of classifying real rank zero C^*-algebras.

0.2 The structure of the thesis

In this thesis we consider a slightly larger inductive limit class than the AF algebras, namely the class of AD algebras which consists of the countable inductive limits of finite direct sums of matrix algebras over the circle algebra and the dimension drop algebras. A class that still seems fairly small but has proven to contain lots of interesting C^*-algebras, including the irrational rotation algebras, cf. [EE93].

As we will see, the above invariant, even with the group $\mathrm{K}_1(-)$ and the positive cone $\mathrm{K}_*(-)^+$ added, is not complete for the class of AD algebras.

Hence K-theory with coefficients in $\mathbb{Z}/n$ is introduced, and this expanded invariant turns out to be strongly complete for the class of real rank zero AD algebras where $\operatorname{tor}\mathrm{K}_1(-)$ is annihilated by n. Via the range result for the invariant that was established in [ET], one sees that the invariant may be reduced to ordinary ordered K-theory when one considers, for example, only simple AD algebras.

Before all this is done, however, we first introduce inductive limits of both C^*-algebras and abelian groups – to be able to properly define the C^*-algebras in question – and then introduce K- and KK-theory – to be able to properly define the invariant in question.

0.3 As for the notation

As for the notation, we write $\mathrm{S}\,A = C_0((0,1), A)$ for the suspension of a C^*-algebra A, $-\otimes-$ denotes the minimal tensor product of C^*-algebras, and $A^\sim$ denotes the minimal unitization of a C^*-algebra A. Also, we denote the $n \times n$ matrices over $\mathbb{C}$ by M_n, and the $n \times n$ unit-matrix by 1_n.

By an ideal in a C^*-algebra we always mean a two-sided, closed ideal. For any unital C^*-algebra A we denote the unit in A by 1, and by $\eta\colon \mathbb{C} \to A$ we denote the canonical embedding $x \mapsto x1$. In all other cases embeddings whose definition seem obvious, will be denoted by ι.

We will be dealing a lot with finite abelian groups and will therefore be doing some integer calculations: by (n, k) we denote the greatest common divisor of the integers n and k, and we write $n \mid k$ when n divides k.

0.4 Thank you for the music

The thesis was completed when I visited the Fields Institute in Toronto, so thanks are due to the Fields Institute for their hospitality and to Niels Bohr Fondet for their financial support of my visit. Finally, I would like to thank my husband Tarje Bargheer for being more perfect than me at being a manifestation of that which I am, my friend David Kyed for bringing me chai and croissants, and my supervisor Søren Eilers in particular for highly professional guidance, for always insisting that I should ask him more questions, and of course for doing such fun mathematics. I would also like to thank all three for providing me with music to listen to while working on this thesis.

Sara Arklint
Nørrebro, December 2007

1 Inductive limits

As the C^*-algebras of interest in this thesis is an inductive limit class, we must first become acquainted with the notion of an inductive limit. To clarify the notation and constructions used throughout the thesis, a short introduction to inductive limits is here given. Consult e.g. [Wei94] and [RLL00] for more details on inductive limits in general respectively for C^*-algebras and (ordered) abelian groups.

1.1 Inductive limits within a general category

First we define the category of inductive I-systems over a fixed category.

Definition 1.1 In a given category an *inductive I-system* $(X_i, f_{i,j})$ over a directed set I consists of objects X_i for each $i \in I$ and morphisms $f_{i,j}\colon X_j \to X_i$ for each $i > j$ that satisfy $f_{i,k} = f_{i,j} f_{j,k}$ when $i > j > k$. A morphism $(\varphi_i)\colon (X_i, f_{i,j}) \to (Y_i, g_{i,j})$ of inductive systems is a family of morphisms $\varphi_i\colon X_i \to Y_i$ that satisfy $\varphi_i f_{i,j} = g_{i,j}\varphi_j$ whenever $i > j$.

Notice that a covariant functor will map an inductive I-system into an inductive I-system and a morphism of systems into a morphism of systems.

We now define the inductive limit of an inductive system. One may think of the limit as a sort of least upper bound of the system, if one thinks of the morphisms $f_{i,j}\colon X_j \to X_i$ as inequalitys $X_j < X_i$.

Definition 1.2 Considering an object X together with a family of morphisms $f_i\colon X_i \to X$, we say that (X, f_i) is *compatible* with the system $(X_i, f_{i,j})$ if $f_j = f_i f_{i,j}$ whenever $i > j$. And (X, f_i) is called an *inductive limit* of $(X_i, f_{i,j})$ if furthermore it has *the universal property* that whenever (Y, g_i) is also compatible with $(X_i, f_{i,j})$ then there exists exactly one morphism $f\colon X \to Y$ such that $g_i = f f_i$ for all $i \in I$.

If the category behaves nicely, every inductive I-system will have an inductive limit. Notice that the universal property ensures uniqueness of the inductive limit of a given system. Hence we can let $\varinjlim(X_i, f_{i,j})$ (for short $\varinjlim X_i$) denote the limit of the system $(X_i, f_{i,j})$ if it exists. The universal property also ensures that a cofinal subsystem will have the same limit as the original system.

We will only be dealing with countable inductive systems, i.e. inductive $\mathbb{N}$-systems, and only within the category of C^*-algebras and within the category of (ordered) abelian groups. We will write the inductive $\mathbb{N}$-systems as (X_i, f_i) with $f_i = f_{i+1,i}$ as one can reconstruct $f_{i,j}$ as $f_{i,j} = f_{i-1} \cdots f_j$ whenever $i > j$. Within these categories any inductive system will admit an inductive limit, and we use the following constructions of the limit.

Please note that to ease notation, representatives of cosets are considered instead of the cosets themselves.

1.2 Inductive limits of C^*-algebras

Construction 1.3 Given an inductive system (A_i, f_i) of C^*-algebras, we consider the C^*-algebra $\prod_{i\in\mathbb{N}} A_i$ of bounded sequences (a_i) with each $a_i \in A_i$, together with its ideal

$\sum_{i\in\mathbb{N}} A_i$ consisting of sequences (a_i) where $\|a_i\|$ converges to zero. For each $i \in \mathbb{N}$ one defines

$$f_{\infty,i}\colon A_i \to \prod_{j\in\mathbb{N}} A_j \Big/ \sum_{j\in\mathbb{N}} A_j$$

as $f_{\infty,i}(a) = (a_j)$ with $a_j = f_{j,i}(a)$ when $j > i$, $a_i = a$, and $a_j = 0$ elsewise. Together with the C^*-subalgebra

$$A = \overline{\bigcup_{i\in\mathbb{N}} \operatorname{im} f_{\infty,i}}$$

these homomorphisms form the inductive limit $(A, f_{\infty,i})$ of the system (A_i, f_i).

In fact, $\varinjlim$ is a functor from the category of inductive systems of C^*-algebras to the category of C^*-algebras, and for a morphism $(\alpha_i)\colon (A_i, f_i) \to (B_i, g_i)$ one defines $\varinjlim \alpha_i \colon \varinjlim A_i \to \varinjlim B_i$ on the above dense subset as $(a_i) \mapsto (\alpha_i(a_i))$.

Consider an inductive system (A_n, f_n) and a (B, β_n) which is compatible with it. According to [RLL00, 6.2.4], (B, β_n) is the limit of (A_n, f_n) if $B = \overline{\bigcup_n \operatorname{im} \beta_n}$ and $\ker \beta_n \subseteq \ker f_{\infty,n}$ holds for all $n \in \mathbb{N}$. This is quite useful when one wants to determine the limit of a concrete inductive system.

Example 1.4 Consider the maps $f_n\colon M_n \to M_{n+1}$ given by $a \mapsto \operatorname{diag}(a, 0)$. This makes (M_n, f_n) an inductive system of C^*-algebras, and we now claim that the compact operators $K(H)$ on the separable Hilbert space $H = \ell^2(\mathbb{N})$ is its limit. Let (e_n) denote an orthonormal basis for H, and define $\varphi_n\colon M_n \to K(H)$ as $\varphi_n(a)(e_k) = \sum_i a_{ik} e_i$ when $a = (a_{ij})$.

Notice that $\varphi_n = \varphi_{n+1} f_n$ for any n, i.e. that $(K(H), \varphi_n)$ is compatible with (M_n, f_n). Clearly, $\ker \varphi_n = 0 \subseteq \ker f_{\infty,n}$ for any n, and as $\bigcup_n \operatorname{im} \varphi_n$ equals the operators of finite rank, $\overline{\bigcup_n \operatorname{im} \varphi_n} = K(H)$. Hence $(K(H), \varphi_n) = \varinjlim(M_n, f_n)$.

1.3 Inductive limits of (ordered) abelian groups

Inductive limits of abelian groups is also of interest to us as the K-groups of a C^*-algebra are abelian groups. Since the K-groups to be defined and studied later on are ordered groups, we first need to know the definition of the category of ordered groups. The category of ordered groups has as objects ordered groups and as morphisms positive group homomorphisms.

Definition 1.5 An *ordered group* (G, G^+) is an abelian group G together with a subset $G^+ \subseteq G$, called *the positive cone* of G, that satisfies $G^+ + G^+ \subseteq G^+$, $G^+ \cap (-G^+) = 0$ and $G^+ + (-G^+) = G$. A group homomorphism $\varphi\colon G \to H$ between ordered groups is called *positive* if $\varphi(G^+) \subseteq H^+$.

We write $g \geq h$ if $g - h \in G^+$, and one can readily verify that this defines a partial order on G. Notice that $G^+ \cap \operatorname{tor} G = 0$, hence a torsion-group cannot be made into an ordered group.

Construction 1.6 For an inductive system (G_i, f_i) of (ordered) abelian groups, one constructs in a very similar fashion the group

$$G = \left\{ (x_i) \in \prod_{i \in \mathbb{N}} G_i \;\middle|\; f_i(x_i) = x_{i+1} \text{ eventually} \right\} \Big/ \left\{ (x_i) \;\middle|\; x_i = 0 \text{ eventually} \right\}$$

and homomorphisms $f_{\infty,i}\colon G_i \to G$ as $f_{\infty,i}(x) = (x_j)$ with $x_j = f_{j,i}(x_i)$ when $j > i$, $x_i = x$, and $x_j = 0$ elsewise. Then $(G, f_{\infty,i})$ will be the inductive limit of (G_i, f_i) within the category of abelian groups. If the groups G_i are ordered groups, and the maps f_i positive group homomorphisms, one defines $G^+ = \bigcup_{i \in \mathbb{N}} f_{\infty,i}(G_i^+)$ hereby ordering G, and $(G, f_{\infty,i})$ will then be the inductive limit of (G_i, f_i) within the category of ordered abelian groups.

Notice that by construction

$$G = \bigcup_{i \in \mathbb{N}} \operatorname{im} f_{\infty,i}.$$

Again, $\varinjlim$ is a functor, and given a morphism $(\varphi_i)\colon (G_i, f_i) \to (H_i, g_i)$ one defines $\varinjlim \varphi_i\colon \varinjlim G_i \to \varinjlim H_i$ as $(x_i) \mapsto (\varphi_i(x_i))$.

Consider an inductive system (G_n, f_n) and a (H, φ_n) which is compatible with it. According to [RLL00, 6.2.5], (H, φ_n) is the limit of (G_n, f_n) if $H = \bigcup_n \operatorname{im} \varphi_n$ and $\ker \varphi_n \subseteq \ker f_{\infty,n}$ holds for all $n \in \mathbb{N}$. Again, this is quite useful when one wants to determine the limit of a concrete inductive system.

Example 1.7 Consider the inductive system $(\mathbb{Z}, 2)$, i.e. the system (G_n, f_n) with $G_n = \mathbb{Z}$ and $f_n\colon \mathbb{Z} \to \mathbb{Z}$ defined as $f_n(x) = 2x$. Consider the group $\mathbb{Z}[\frac{1}{2}]$, i.e. the subgroup $\{\frac{x}{2^k} \mid x \in \mathbb{Z}, k \in \mathbb{N}_0\}$ of $\mathbb{Q}$, and define $\varphi_n\colon \mathbb{Z} \to \mathbb{Z}[\frac{1}{2}]$ as $\varphi_n(x) = \frac{x}{2^n}$.

Now, $\varphi_n = \varphi_{n+1} f_n$ for any n, hence $(\mathbb{Z}[\frac{1}{2}], \varphi_n)$ is compatible with $(\mathbb{Z}, 2)$. Notice that $\ker \varphi_n = 0 \subseteq \ker f_{\infty,n}$ for each n; and clearly $\bigcup_n \operatorname{im} \varphi_n = \mathbb{Z}[\frac{1}{2}]$. Hence, $(\mathbb{Z}[\frac{1}{2}], \varphi_n) = \varinjlim(\mathbb{Z}, 2)$.

2 K-theory and KK-theory

Later on we will define an invariant for AD algebras, and it will be defined in means of KK-theory. The approach in this thesis to K-theory and KK-theory has been on a strictly need-to-know basis. This section is meant as a not-to-rough introduction to the necessary parts of the subjects, mostly aimed at those readers who are not completely foreign to K-theory. The main sources have been [Bla98] and [RLL00].

We will be regarding K-theory both as a special case of KK-theory, and, when necessary, as formal differences of equivalence classes of projections resp. homotopy classes of unitaries. We use the more-or-less standard notation of $[p]_0$ denoting the equivalence class in $\mathrm{K}_0(A)$ of the projection p, $[u]_1$ the homotopy class in $\mathrm{K}_1(A)$ of the unitary u, and $[\alpha]$ the KK-class in $\mathrm{KK}(A, B)$ represented by the $*$-homomorphism $\alpha\colon A \to B$.

2.1 An idea of $\mathrm{KK}(A, B)$

$\mathrm{KK}(-,-)$ is a bifunctor, contravariant in the first variable and covariant in the second, from the category of C^*-algebras to the category of abelian groups (sometimes into the category of ordered abelian groups).

For any $k \in \mathbb{N}$, a $*$-homomorphism $\alpha\colon A \to M_k(B)$ induces a KK-class, i.e. an element $[\alpha]$ in $\mathrm{KK}(A, B)$, and two homotopic $*$-homomorphisms lie in the same KK-class. By Cuntz' Quasihomomorphism Picture of $\mathrm{KK}(A, B)$, one may think of the elements in $\mathrm{KK}(A, B)$ as homotopy classes of generalised $*$-homomorphisms. In fact, most of the KK-groups we consider are generated by formal differences of classes that are represented by $*$-homomorphisms. As noted in [Eil95, p. 19],

$$\mathrm{KK}(\mathbb{I}_n, A) = \{[\alpha] \mid \alpha\colon \mathbb{I}_n \to M_k(A) \text{ a } *\text{-homomorphism}\}$$

and

$$\mathrm{KK}(\mathbb{I}_n^\sim, A) = \{[\alpha] - [\alpha'] \mid \alpha, \alpha'\colon \mathbb{I}_n \to M_k(A) \text{ are } *\text{-homomorphisms}\},$$

where the C^*-algebras $\mathbb{I}_n$ and $\mathbb{I}_n^\sim$ are to be defined in section 3, and the groups $\mathrm{KK}(\mathbb{I}_n, A)$ and $\mathrm{KK}(\mathbb{I}_n^\sim, A)$ are to be studied in section 4.

2.2 Some isomorphisms

The natural isomorphisms mentioned below will be regarded as identities. See [RS87, 1.11] for a reference.

Recall that two C^*-algebras A and B are said to be *homotopy equivalent* if there are $*$-homomorphisms $\alpha\colon A \to B$ and $\beta\colon B \to A$ such that $\beta\alpha$ is homotopic to id_A (i.e. such that there is a point-wise continuous path of $*$-homomorphisms from $\beta\alpha$ to id_A) and such that $\alpha\beta$ is homotopic to id_B.

Proposition 2.1 The functors $\mathrm{KK}(\mathbb{C}, -)$ and $\mathrm{KK}(\mathrm{S}\,\mathbb{C}, -)$ are naturally isomorphic to $\mathrm{K}_0(-)$ respectively $\mathrm{K}_1(-)$; the first isomorphism given by $[p]_0 \mapsto [1 \mapsto p]$. The bifunctor $\mathrm{KK}(-,-)$ is homotopy invariant and stably invariant in both variables; the stability isomorphism being induced by $a \mapsto \mathrm{diag}(a, 0, \ldots)$. One has Bott periodicity, i.e. that $\mathrm{KK}(\mathrm{S}\,-,-)$ is naturally isomorphic to $\mathrm{KK}(-, \mathrm{S}\,-)$, and $\mathrm{KK}(\mathrm{S}^2\,-,-)$ to $\mathrm{KK}(-,-)$.

In some situations we will need the explicit construction of the Bott map from $\mathrm{K}_0(A)$ to $\mathrm{K}_1(\mathrm{S}\,A)$, so we state it here. See [RLL00, 11.1-2] for more details and a proof.

Proposition 2.2 Given a C^*-algebra A, *the Bott map*

$$\beta_A\colon\ \mathrm{K}_0(A) \to \mathrm{K}_1(\mathrm{S}\,A)$$

is an isomorphism. When A is unital, β_A is defined by $[p]_0 \mapsto [f_p]_1$ where $f_p(t) = e^{2\pi i t}p + (1_n - p)$ when $p \in M_n(A)$.

2.3 The Kasparov product

The following map, the *Kasparov product*, is quite essential. Again, see [RS87, 1.11] and [Bla98, 18.7.1] for references.

Proposition 2.3 Given C^*-algebras A, B and C, there is a map

$$\mathrm{KK}(A,B) \times \mathrm{KK}(B,C) \to \mathrm{KK}(A,C)$$

which is associative, distributive with respect to the group structures, and natural in both variables. When $A = B$ then id_A is a left unit, and when $B = C$ then id_B is a right unit. It generalizes composition of $*$-homomorphisms, i.e. given $*$-homomorphisms $\alpha\colon A \to B$ and $\beta\colon B \to C$ then $[\beta\alpha] = [\alpha][\beta]$.

For an element $x \in \mathrm{KK}(A,B)$ we let $x\cdot\colon \mathrm{KK}(B,C) \to \mathrm{KK}(A,C)$ denote the group homomorphism given by Kasparov multiplication from the left $y \mapsto xy$, and we let $\cdot x\colon \mathrm{KK}(C,A) \to \mathrm{KK}(C,B)$ denote the group homomorphism given by Kasparov multiplication from the right $y \mapsto yx$.

A family of group homomorphisms $\varphi_A\colon \mathrm{KK}(A,B) \to \mathrm{KK}(A,C)$ is called KK-*natural* (resp. *natural*) in A if the diagram

$$\begin{array}{ccc} \mathrm{KK}(A,B) & \xrightarrow{\varphi_A} & \mathrm{KK}(A,C) \\ {\scriptstyle x\cdot}\uparrow & & \uparrow{\scriptstyle x\cdot} \\ \mathrm{KK}(A',B) & \xrightarrow{\varphi_{A'}} & \mathrm{KK}(A',C) \end{array}$$

commutes for all A and A' and all $x \in \mathrm{KK}(A,A')$ (resp. for all $x \in \mathrm{KK}(A,A')$ of the form $x = [\alpha]$ with $\alpha\colon A \to A'$ a $*$-homomorphism). The term KK-naturality makes even more sense if one thinks of it as naturality within the KK-category whose objects are separable C^*-algebras and whose morphisms from A to B are $\mathrm{KK}(A,B)$. KK-naturality and naturality of a family of maps $\psi_B\colon \mathrm{KK}(A,B) \to \mathrm{KK}(D,B)$ is defined similarly.

For nicely behaving C^*-algebras one has the six-term exact sequences. See e.g. [RS87, 1.11] for a reference.

Proposition 2.4 When B and D are separable C^*-algebras and B is nuclear, then any short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of C^*-algebras induces two exact sequences

$$\begin{array}{ccccc}
\mathrm{KK}(D,A) & \xrightarrow{\cdot[\alpha]} & \mathrm{KK}(D,B) & \xrightarrow{\cdot[\beta]} & \mathrm{KK}(D,C) \\
\delta_1 \uparrow & & & & \downarrow \delta_0 \\
\mathrm{KK}(\mathrm{S}\,D,C) & \xleftarrow[\cdot[\beta]]{} & \mathrm{KK}(\mathrm{S}\,D,B) & \xleftarrow[\cdot[\alpha]]{} & \mathrm{KK}(\mathrm{S}\,D,A)
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{KK}(A,D) & \xleftarrow{[\alpha]\cdot} & \mathrm{KK}(B,D) & \xleftarrow{[\beta]\cdot} & \mathrm{KK}(C,D) \\
\delta^1 \downarrow & & & & \uparrow \delta^0 \\
\mathrm{KK}(C,\mathrm{S}\,D) & \xrightarrow[{[\beta]\cdot}]{} & \mathrm{KK}(B,\mathrm{S}\,D) & \xrightarrow[{[\alpha]\cdot}]{} & \mathrm{KK}(A,\mathrm{S}\,D)
\end{array}$$

that are KK-natural in D.

The vertical connecting maps of the two six-term exact sequences associated to a mapping cone sequence are easy to describe. See [Bla98, 19.4.3].

Definition 2.5 The *mapping cone sequence* associated to the map $\alpha\colon A \to B$ is the short-exact sequence

$$0 \longrightarrow \mathrm{S}\,B \xrightarrow{\iota} \mathrm{C}(\alpha) \xrightarrow{\pi} A \longrightarrow 0$$

where the cone of α is defined as

$$\mathrm{C}(\alpha) = \{(a,f) \in A \oplus C_0((0,1],B) \mid f(1) = \alpha(a)\}$$

and the maps are given as $\iota(f) = (0,f)$ and $\pi(a,f) = a$.

Proposition 2.6 Given a map $\alpha\colon A \to B$ between nuclear, separable C^*-algebras A and B, and a separable C^*-algebra D, then, using Bott periodicity, the vertical connecting maps of the six-term exact sequences associated to the mapping cone sequence of α, are in the covariant case

$$\mathrm{KK}(D,A) \xrightarrow{\cdot[\alpha]} \mathrm{KK}(D,B) = \mathrm{KK}(\mathrm{S}\,D,\mathrm{S}\,B)$$

and

$$\mathrm{KK}(\mathrm{S}\,D,A) = \mathrm{KK}(D,\mathrm{S}\,A) \xrightarrow{\cdot[\mathrm{S}\,\alpha]} \mathrm{KK}(D,\mathrm{S}\,B)\ ,$$

and in the contravariant case

$$\mathrm{KK}(\mathrm{S}\,B,\mathrm{S}\,D) = \mathrm{KK}(B,D) \xrightarrow{[\alpha]\cdot} \mathrm{KK}(A,D)$$

and

$$\mathrm{KK}(\mathrm{S}\,B,D) \xrightarrow{[\mathrm{S}\,\alpha]\cdot} \mathrm{KK}(\mathrm{S}\,A,D) = \mathrm{KK}(A,\mathrm{S}\,D)\ .$$

2.4 Order structure on $\mathrm{KK}(A,B)$

We desire to define K-theory with coefficients via KK-theory, and as we desire to define an order on these new K-groups, we need the following definition.

Definition 2.7 Define *the positive cone* as

$$\mathrm{KK}(A,B)^+ = \{[\alpha] \mid \alpha\colon A \to M_k(B) \text{ a } *\text{-homomorphism }, k \in \mathbb{N}\} \subseteq \mathrm{KK}(A,B),$$

and *the scale* as

$$\mathrm{KK}(A,B)^{+,\Sigma} = \{[\alpha] \mid \alpha\colon A \to B \text{ a } *\text{-homomorphism }\} \subseteq \mathrm{KK}(A,B).$$

In the general case, this does not make $(\mathrm{KK}(A,B), \mathrm{KK}(A,B)^+)$ into an ordered group. However, via the isomorphism $\mathrm{K}_0(-) = \mathrm{KK}(\mathbb{C},-)$ of 2.1, it is a generalization of the positive cone

$$\mathrm{K}_0(A)^+ = \{[p]_0 \mid p \in M_n(A) \text{ a projection }, n \in \mathbb{N}\}$$

and the scale

$$\Sigma(A) = \{[p]_0 \mid p \in A \text{ a projection}\}$$

in $\mathrm{K}_0(A)$.

2.5 Split-exactness

Recall that a short sequence of $\mathbb{Z}$-modules or C^*-algebras

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is said to *split* if there are *splitting maps* $\sigma\colon C \to B$ and $\tau\colon B \to A$ satisfying $\tau\varphi = \mathrm{id}_A$, $\psi\sigma = \mathrm{id}_C$ and $\varphi\tau + \sigma\psi = \mathrm{id}_B$. Since a short sequence that splits is exact, we also refer to such sequences as *split-exact*. If the above sequence is exact and we have a map $\sigma\colon C \to B$ with $\psi\sigma = \mathrm{id}_C$, we can construct one and only one map $\tau\colon B \to A$ such that τ and σ are splitting maps and the sequence in question splits, and we will therefore often only specify the map σ and refer to it as *the splitting map*.

Lemma 2.8 Given any split-exact sequence

$$0 \longrightarrow A \underset{\tau}{\overset{\varphi}{\rightleftarrows}} B \underset{\sigma}{\overset{\psi}{\rightleftarrows}} C \longrightarrow 0$$

of nuclear, separable C^*-algebras and any separable C^*-algebra D, the two induced sequences in KK-theory

$$0 \longrightarrow \mathrm{KK}(C,D) \underset{[\sigma]\cdot}{\overset{[\psi]\cdot}{\rightleftarrows}} \mathrm{KK}(B,D) \underset{[\tau]\cdot}{\overset{[\varphi]\cdot}{\rightleftarrows}} \mathrm{KK}(A,D) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{KK}(D,A) \underset{\cdot[\tau]}{\overset{\cdot[\varphi]}{\rightleftarrows}} \mathrm{KK}(D,B) \underset{\cdot[\sigma]}{\overset{\cdot[\psi]}{\rightleftarrows}} \mathrm{KK}(D,C) \longrightarrow 0$$

are both split-exact with splitting maps induced by the splitting maps of the sequence of C^*-algebras, and the induced maps are all positive. The splittings are KK-natural in D.

Proof. As D is separable we get by 2.4 that $\mathrm{KK}(-,D)$ and $\mathrm{KK}(D,-)$ are half-exact on short-exact sequences of nuclear C^*-algebras, and it is easily seen that any half-exact co- or contravariant functor is split-exact. The definition of the positive cone in the KK-groups insures that the induced maps are positive by construction. And the splittings are KK-natural in D by associativity of the Kasparov product. ♡

Remark 2.9 Consider two homomorphisms $\varphi, \psi \colon A \to B$ where $A = A_1 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus \cdots \oplus B_n$, and denote by $\iota_s^A \colon A_s \to A$, $\pi_s^A \colon A \to A_s$, $\iota_r^B \colon B_r \to B$ and $\pi_r^B \colon B \to B_r$ the canonical injections and projections. As a consequence of lemma 2.8

$$\sum_{s=1}^{m} [\iota_s^A \pi_s^A] = [\mathrm{id}_A] \quad \text{and} \quad \sum_{r=1}^{n} [\iota_r^B \pi_r^B] = [\mathrm{id}_B],$$

hence if $[\pi_r^B \varphi \iota_s^A] = [\pi_r^B \psi \iota_s^A]$ for all r, s then

$$\begin{aligned}
[\varphi] &= [\mathrm{id}_A][\varphi][\mathrm{id}_B] \\
&= \sum_{r,s} [\iota_s^A \pi_s^A][\varphi][\iota_r^B \pi_r^B] \\
&= \sum_{r,s} [\pi_s^A][\pi_r^B \varphi \iota_s^A][\iota_r^B] \\
&= \sum_{r,s} [\pi_s^A][\pi_r^B \psi \iota_s^A][\iota_r^B] \\
&= [\psi]
\end{aligned}$$

as elements of $\mathrm{KK}(A,B)$.

2.6 The Universal Coefficient Theorem

Since we will have some use of the Universal Coefficient Theorem by Rosenberg and Schochet, we will be needing the notion of the bootstrap class.

Definition 2.10 The *bootstrap class* is the smallest class of C^*-algebras that contains the separable, abelian C^*-algebras and is closed under stable isomorphism, countable inductive limits, extensions, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$.

As a direct sum is an extension, we notice that the bootstrap class is closed under taking countable direct sums.

When considering the graded K_*-groups, we abuse the notation $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$ by letting it denote the graded homomorphisms $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$; the same applies to $\mathrm{Ext}^1(K_*(A), \mathrm{K}_*(B))$ as we let it denote $\mathrm{Ext}^1(\mathrm{K}_0(A), \mathrm{K}_0(B)) \oplus \mathrm{Ext}^1(\mathrm{K}_1(A), \mathrm{K}_1(B))$.

Theorem 2.11 (Universal Coefficient Theorem, [RS87]) If the C^*-algebra A is separable and nuclear and lies in the bootstrap class, and the C^*-algebra B is σ-unital, then there exists a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(K_*(A), K_*(\mathrm{S}\, B)) \xrightarrow{\Delta} \mathrm{KK}(A, B) \xrightarrow{\Gamma} \operatorname{Hom}(\mathrm{K}_*(A), K_*(B)) \longrightarrow 0$$

which is KK-natural in A and B. The map Γ is given by Kasparov multiplication from the right. The sequence splits unnaturally.

The definition of Γ uses the identifications $\mathrm{K}_*(A) = \mathrm{KK}(\mathbb{C}, A) \oplus \mathrm{KK}(\mathrm{S}\,\mathbb{C}, A)$ and $\mathrm{K}_*(B) = \mathrm{KK}(\mathbb{C}, B) \oplus \mathrm{KK}(\mathrm{S}\,\mathbb{C}, B)$; so given $a \in \mathrm{KK}(A, B)$, $\Gamma(a)$ is the map $(x, y) \mapsto (xa, ya)$ where xa and ya denotes the Kasparov product of x with a and y with a.

2.7 Continuity of $\mathrm{KK}(A, -)$

As we are dealing with an inductive limit class, the following is indispensable. Via [Bla98, 21.3.1] it follows from [RS87, 7.13].

Proposition 2.12 If A is separable, nuclear and lie in the bootstrap class, and the group $\mathrm{K}_*(A)$ is finitely generated, then $\mathrm{KK}(A, -)$ is continuous, i.e. given an inductive system (B_i, f_i) of C^*-algebras such that $\varinjlim(B_i, f_i)$ is σ-unital then $(\mathrm{KK}(A, \varinjlim B_i), \cdot[f_{\infty,i}])$ is naturally isomorphic to the limit of the inductive system $(\mathrm{KK}(A, B_i), \cdot[f_i])$ of abelian groups. If furthermore each $\mathrm{KK}(A, B_i)$ is an ordered group, then $(\mathrm{KK}(A, \varinjlim B_i), \cdot[f_{\infty,i}])$ is the limit of $(\mathrm{KK}(A, B_i), \cdot[f_i])$ within the category of ordered groups.

As for the scale, one sees by reading the proof of [RLL00, 6.3.2] that in the case of $A = \mathbb{C}$ then $\mathrm{KK}(\mathbb{C}, B)^{+,\Sigma} = \bigcup_i \{[f_i \alpha] \mid \alpha \colon \mathbb{C} \to B_i\}$.

3 AD algebras

We are now ready to define the algebras in question and then show that they satisfy certain propertys. We will also state some quite powerful technical theorems that we will be needing later on.

3.1 Definition of AD algebras

Consider for each $n \in \mathbb{N}\backslash\{1\}$ the nonunital dimension drop algebra

$$\mathbb{I}_n = \{f \in C([0,1], M_n) \mid f(0) = 0, f(1) \in \mathbb{C}1_n\}$$

and the dimension drop algebra

$$\mathbb{I}_n^{\sim} = \{f \in C([0,1], M_n) \mid f(0), f(1) \in \mathbb{C}1_n\}$$

where 1_n denotes the identity matrix in the $n \times n$ matrices M_n.

To get a feeling of the nonunital dimension drop algebras, we start out with a small technical lemma from [DL94, 1.7].

Lemma 3.1 Consider a $*$-homomorphism $\alpha\colon \mathbb{I}_n \to A$ for some C^*-algebra A. Then the $*$-homomorphism $\mathrm{diag}(\alpha, \ldots, \alpha)\colon \mathbb{I}_n \to M_n(A)$ is null homotopic.

Proof. Define $\beta\colon \mathbb{I}_n \to M_n(\mathbb{I}_n)$ as $f \mapsto \mathrm{diag}(f, \ldots, f)$. As $\mathrm{diag}(\alpha, \ldots, \alpha)$ factors through β, it suffices to prove that β is null homotopic.

View $M_n(\mathbb{I}_n)$ as $\{f \in C([0,1], M_n \otimes M_n \mid f(0) = 0, f(1) \in \mathbb{C}(M_n \otimes 1_n)\}$, whereby $\beta(f) = 1_n \otimes f$, and let $u \in M_n \otimes M_n$ denote a unitary satisfying $u(1_n \otimes M_n)u^* = M_n \otimes 1_n$. Now, define a point-wise continuous path $s \mapsto \beta_s\colon \mathbb{I}_n \to M_n(\mathbb{I}_n)$ as

$$\beta_s(f)(t) = u(1_n \otimes f(st))u^*.$$

As $\beta_0 = 0$ and $\beta_1 = \mathrm{ad}(u)\beta$, we see that $\mathrm{ad}(u)\beta$ is null homotopic. And by taking a point-wise continuous path $s \mapsto u_s \in M_n \otimes M_n$ with $u_0 = u$ and $u_1 = 1$, we get a path $s \mapsto \mathrm{ad}(u_s)\beta$ from $\mathrm{ad}(u)\beta$ to β. ♡

For later use, we define $\delta_i\colon \mathbb{I}_n^{\sim} \to \mathbb{C}$ by $\delta_i(f)1_n = f(i)$, a sort of evaluation at the endpoint $i \in \{0,1\}$. We also define the maps $\mathrm{in}_{n,m}, \overline{\mathrm{in}}_{n,m}\colon \mathbb{I}_m^{\sim} \to M_{\frac{m}{(n,m)}}(\mathbb{I}_n^{\sim})$ as follows. Given $f \in \mathbb{I}_m^{\sim}$, we consider $\mathrm{diag}(f, \ldots, f) \in M_{\frac{nm}{(n,m)}}(C([0,1]))$ consisting of $\frac{n}{(n,m)}$ copies of f on the diagonal. As $\mathrm{diag}(f(0), \ldots, f(0)), \mathrm{diag}(f(1), \ldots, f(1)) \in \mathbb{C}1_{\frac{nm}{(n,m)}}$, one may view $\mathrm{diag}(f, \ldots, f)$ as an element of $M_{\frac{m}{(n,m)}}(\mathbb{I}_n^{\sim})$. We define $\mathrm{in}_{n,m}(f) = \mathrm{diag}(f, \ldots, f)$ and $\overline{\mathrm{in}}_{n,m}(f)(t) = \mathrm{diag}(f(1-t), \ldots, f(1-t))$.

The AD algebras are build from the the circle algebra $C(S^1)$ and the unital dimension drop algebras $\mathbb{I}_n^{\sim}$ by being the smallest class of C^*-algebras that contains $C(S^1)$ and $\mathbb{I}_2^{\sim}, \mathbb{I}_3^{\sim}, \mathbb{I}_4^{\sim}, \ldots$ and is closed under tensoring with M_n, under taking finite direct sums, and under taking countable inductive limits.

Definition 3.2 An AD algebra is a C^*-algebra isomorphic to a countable inductive limit of the form

$$\varinjlim \left(\bigoplus_{k=1}^{N_n} M_{m_{n,k}}(A_{n,k}), f_n \right)$$

with each $A_{n,k} \in \{C(S^1), \mathbb{I}_2^\sim, \mathbb{I}_3^\sim, \mathbb{I}_4^\sim, \ldots\}$.

We will refer to $C(S^1)$ and $\mathbb{I}_2^\sim, \mathbb{I}_3^\sim, \mathbb{I}_4^\sim, \ldots$ as *the building blocks.*

As for the name AD algebras, it stands for approximately dimension drop algebra (regardless the presence of the circle algebra among the building blocks), and it is inspired by the abbreviation AF algebra for approximately finite C^*-algebra. The AF algebras being the countable inductive limits of finite dimensional C^*-algebras, i.e. of finite direct sums of matrix algebras. Same applies for the naming AT algebras, or approximately torus algebras, of the class of countable inductive limits of finite direct sums of matrix algebras over the circle algebra $C(S^1)$.

As we will see, the class of AF algebras is contained in the class of AD algebras. One could then count $\mathbb{C}$ among the building blocks, but as some of our more technical lemmata are proven for one type of building block at a time, we save time and effort by not doing so.

Lemma 3.3 The class of AF algebras is contained in the class of AD algebras.

Proof. Given an AF algebra $A = \varinjlim(A_i, f_i)$ with each $A_i = M_{n_{i,1}} \oplus \cdots \oplus M_{n_{i,k_i}}$, we define for each i a C^*-algebra

$$B_i = C(S^1, A_i) = M_{n_{i,1}}(C(S^1)) \oplus \cdots \oplus M_{n_{i,k_i}}(C(S^1))$$

and $*$-homomorphisms $\alpha_i \colon A_i \to B_i, a \mapsto (t \mapsto a)$ and $\beta_i \colon B_i \to A_i, b \mapsto b(x_0)$ where $x_0 \in S^1$ is a fixed point on the circle.

By defining $g_i = \alpha_{i+1} f_i \beta_i$ we make (B_i, g_i) into an inductive system with $B = \varinjlim(B_i, g_i)$ being an AD algebra by definition. As $\beta_i \alpha_i = \mathrm{id}_{A_i}$ for each i, one sees that $g_i \alpha_i = \alpha_{i+1} f_i$ and $f_i \beta_i = \beta_{i+1} g_i$, i.e. that (α_i) and (β_i) are morphisms of inductive systems.

Since $\beta_i \alpha_i = \mathrm{id}_{A_i}$ for each i, $\varinjlim \beta_i \varinjlim \alpha_i = \mathrm{id}_A$. And as

$$\bigcup_{i \in \mathbb{N}} \operatorname{im} g_{\infty,i} = \{(b_i) \mid g_i(b_i) = b_{i+1} \text{ eventually}\}$$

is dense in B, and as $\alpha_{i+1}\beta_{i+1} g_i = g_1$, $\varinjlim \alpha_i \varinjlim \beta_i = \mathrm{id}_B$. Whereby we conclude that $A \cong B$. ♡

3.2 Properties of AD algebras

As we will see, the AD algebras are separable, nuclear and of stable rank one, and lie in the bootstrap class. They are not all of real rank zero, however, and we will define the notion of an inductive system having large denominators to describe what it means for an AD algebra to have real rank zero.

Recall that a C^*-algebra A is said to be *nuclear* if the algebraic tensor product of A with any C^*-algebra B can be equipped with but one C^*-norm. The class of nuclear C^*-algebras is closed under direct sums, extensions, inductive limits, tensor products and quotients, cf. e.g. [Bla06, II.9.4.5]. According to a theorem by M. Takesaki all abelian C^*-algebras are nuclear, cf. e.g. [Mur90, 6.4.15]. We will only be dealing with nuclear C^*-algebras.

Recall also that a unital C^*-algebra A is said to be of *stable rank one* if the set A^{-1} of invertibles in A is dense in A, and a nonunital C^*-algebra A is of stable rank one if its minimal unitization $A^\sim$ is. Using that the left or right invertible elements of a unital stable rank one C^*-algebra are invertible, cf. [Bla06, V.3.1.5], one can easily show that a unital stable rank one C^*-algebra is finite, and as according to [Bla06, V.3.1.16] the matrix algebra $M_n(A)$ is of stable rank one when A is, one sees that any unital stable rank one C^*-algebra is stably finite. This is important to us as any stably finite C^*-algebra A satisfies that $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is an ordered group, cf. [Bla06, V.2.4.8].

Lemma 3.4 All AD algebras, as well as the nonunital dimension drop algebras $\mathbb{I}_n$, $n \in \mathbb{N}\backslash\{1\}$, are nuclear and separable, are of stable rank one, and lie in the bootstrap class.

Proof. As S^1 and $[0,1]$ are metrizable and compact, $C(S^1)$ and $C([0,1])$ are separable according to [Con90, V.6.6], so the building blocks $C(S^1)$ and $\mathbb{I}_n^\sim$ as well as $\mathbb{I}_n$ are all separable. Hence also finite direct sums of matrices over these are separable, and as countable inductive limits of separable C^*-algebras are separable, all AD algebras are separable.

Since the matrix algebras M_m are nuclear, it sufficies to show that the building blocks $C(S^1)$ and $\mathbb{I}_n^\sim$ are nuclear to prove that all AD algebras are nuclear. The circle algebra $C(S^1)$ is nuclear because it is abelian. Consider the extension

$$0 \longrightarrow M_n(C_0((0,1))) \xrightarrow{\iota} \mathbb{I}_n \xrightarrow{\delta_1} \mathbb{C} \longrightarrow 0\,.$$

As $M_n(C_0((0,1))) = M_n \otimes C_0((0,1))$ and $\mathbb{C}$ are nuclear, $\mathbb{I}_n$ is nuclear as well. Considering the extension

$$0 \longrightarrow \mathbb{I}_n \xrightarrow{\iota} \mathbb{I}_n^\sim \xrightarrow{\delta_0} \mathbb{C} \longrightarrow 0$$

we see that also $\mathbb{I}_n^\sim$ is nuclear.

Being separable and abelian, $C_0((0,1))$, $C(S^1)$ and $\mathbb{C}$ lie in the bootstrap class. Since $M_n(C_0((0,1)))$ is stably isomorphic to $C_0((0,1))$, we see that $M_n(C_0((0,1)))$ and thereby also its extension $\mathbb{I}_n$ by $\mathbb{C}$ lie in the bootstrap class. Similarly also $\mathbb{I}_n^\sim$ lies there. As the bootstrap class is closed under stable isomorphism, direct sums and inductive limits, it follows that all AD algebras lie there.

To see that $C(S^1)$ has stable rank one, we notice that any $f \in C(S^1)$ can be approximated by polynomials, and as a polynomial has finitely many roots, the polynomials can be approximated by functions $f \in C(S^1)$ satisfying $f(S^1) \not\ni 0$. Similarly one sees that $C([0,1])$ has stable rank one, so considering the extension

$$0 \longrightarrow \mathrm{S}\,M_n \longrightarrow \mathbb{I}_n^\sim \xrightarrow{\delta_0 \oplus \delta_1} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0$$

where the map $\delta_0 \oplus \delta_1$ admits lifts of unitals we conclude as the C^*-algebra to the left and the C^*-algebra to the right have stable rank one, that also the middle C^*-algebra $\mathbb{I}_n^\sim$ has stable rank one, cf. [Rie83]. Clearly, an inductive limit of stable rank one C^*-algebras is again of stable rank one, and as any matrix algebra over a stable rank one C^*-algebra is of stable rank one, cf. [Bla06, V.3.1.16], we conclude that all AD algebras are of stable rank one. ♡

Now, recall that a unital C^*-algebra A is said to be of *real rank zero* if the set A_{sa}^{-1} of invertible selfadjoints is dense in the set A_{sa} of selfadjoints, and that a nonunital C^*-algebra A is of real rank zero if its minimal unitization $A^\sim$ is. For instance, the direct sum of two real rank zero C^*-algebras is again of real rank zero, and according to [Bla06, V.3.2.10], real rank zero passes to matrix algebras and is preserved in inductive limits, so as $\mathbb{C}$ is of real rank zero, any AF algebra is.

Example 3.5 The building blocks $C(S^1)$ and $\mathbb{I}_n^\sim$ are not of real rank zero as the spaces S^1 and $[0,1]$ are connected. Consider e.g. the selfadjoint $f \in C(S^1)$ given by $f(x) = x + \overline{x}$. Clearly, $f(S^1) = [-2,2]$, and given any invertible selfadjoint $g \in C(S^1)$, $g(S^1) \subseteq \mathbb{R}$ and $0 \notin g(S^1)$, so either $g(S^1) \subseteq \mathbb{R}_+$ or $g(S^1) \subseteq \mathbb{R}_-$ as $g(S^1)$ is connected, hence $\|f-g\|_\infty \geq 2$. As for $\mathbb{I}_n^\sim$, assuming real rank zero we may conclude by [Bla06, V.3.2.9] that the hereditary subalgebra $\mathbb{I}_n$ of $\mathbb{I}_n^\sim$ has an approximate unit of projections, but $\mathbb{I}_n$ has no non-zero projections.

As the building blocks aren't of real rank zero, one may think that the AF algebras are the only real rank zero AD algebras. But, as we will see, there are a lot of real rank zero non-AF AD algebras about, cf. 3.9.

To describe how it affects the connecting maps of its inductive system if an AD algebra is of real rank zero, we will be needing the following definition. Keep in mind that, as $\mathrm{K}_*(C(S^1)) = \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{K}_*(\mathbb{I}_n^\sim) = \mathbb{Z} \oplus \mathbb{Z}/n$, then when considering building blocks A and B, any graded map from $\mathrm{K}_*(A)$ to $\mathrm{K}_*(B)$ is of the form $(x,y) \mapsto (ax, by)$ with $a, b \in \mathbb{Z}$.

Definition 3.6 Given finite direct sums $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_m$ of matrices A_s, B_r over building blocks, we define the set $\mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))^N$ of *N-large* graded homomorphisms $(\varphi_0, \varphi_1)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ as those graded homomorphisms where the maps $\mathrm{K}_0(\pi_r)\varphi_0\,\mathrm{K}_0(\iota_s)\colon x \mapsto a_{rs}x$ and $\mathrm{K}_1(\pi_r)\varphi_1\,\mathrm{K}_1(\iota_s)\colon x \mapsto b_{rs}x$ satisfy (for each pair r,s) that $a_{rs} \geq N$ when $b_{rs} \neq 0$. Here $\iota_s\colon A_s \to A$ denote the canonical inclusion and $\pi_j\colon B \to B_j$ the canonical projection.

Considering an inductive system (A_n, f_n), $(\mathrm{K}_*(A_n), \mathrm{K}_*(f_n))$ is said to have *large denominators* if for any N one can find a subsystem $(A_{n_i}, f_{n_{i+1},n_i})$ of (A_n, f_n) with each $\mathrm{K}_*(f_{n_{i+1},n_i})$ being N-large.

As the concept of having large denominators is somewhat strange, we consider a few simple examples.

Example 3.7 Consider the inductive system $(C(S^1), \mathrm{id})$. The induced system on K-theory is $(\mathbb{Z} \oplus \mathbb{Z}, (1,1))$, and as any subsystem of it will be $(\mathbb{Z} \oplus \mathbb{Z}, (1,1))$, this system does not have large denominators.

Let $\delta\colon C(S^1) \to \mathbb{C}$ denote $f \mapsto f(1)$, and consider the inductive system of C^*-algebras $(M_{2n}(C(S^1)), \mathrm{diag}(\mathrm{id}, \eta\delta))$ which induces the inductive system of groups $(\mathbb{Z} \oplus \mathbb{Z}, (2,1))$. Given an $N \in \mathbb{N}$, we have that $2^N \geq N$, hence we see that the subsystem $(M_{n2^N}(C(S^1)), \mathrm{diag}(\mathrm{id}, \eta\delta, \dots, \eta\delta)$ is N-large. So the system $(\mathbb{Z} \oplus \mathbb{Z}, (2,1))$ has large denominators.

The following result is from [Ell93, 7.2] and will be of great use to us. Notice that it is a direct consequence of the proposition that the building blocks aren't of real rank zero.

Proposition 3.8 Consider an AD algebra $A = \varinjlim(A_n, f_n)$ where each A_n of the system is a finite direct sums of matrices over building blocks. If A is of real rank zero, then $(\mathrm{K}_*(A_n), \mathrm{K}_*(f_n))$ has large denominators.

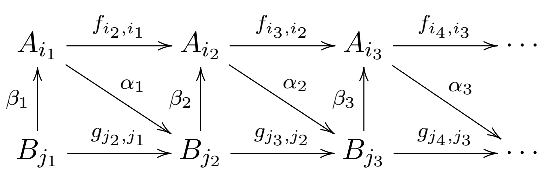
The system af K_*-groups having large denominators doesn't imply that the limit of the system has real rank zero, however. But it is almost true. The following result is also from [Ell93, 6.2], and it will come in handy when we desire to construct certain examples of real rank zero AT and AD algebras.

Proposition 3.9 Consider an inductive system (A_n, f_n) with each A_n being a finite direct sum of matrices over building blocks. If $(\mathrm{K}_*(A_n), \mathrm{K}_*(f_n))$ has large denominators, then one can find a system (A_{n_k}, g_k) which is KK-shape equivalent to (A_n, f_n) and where $\varinjlim(A_{n_k}, g_k)$ has real rank zero.

3.3 KK-shape equivalence

The standard strategy when trying to construct isomorphisms between elements of an inductive limit class, is to make something similar to *Elliott's intertwining argument* work. G. Elliott himself has done so in [Ell93, 7.1], cf. 3.12, and to state (and later on use) his result, we need the notion of KK-shape equivalence.

Definition 3.10 Inductive systems of C^*-algebras (A_i, f_i) and (B_i, g_i) are called KK-*shape equivalent* if there exist subsystems (A_{i_n}, f_{i_n}) and (B_{j_n}, g_{j_n}) and $*$-homomorphisms $\alpha_n\colon A_{i_n} \to B_{j_{n+1}}$ and $\beta_n\colon B_{j_n} \to A_{i_n}$ such that the diagram



commutes at the level of KK-theory, i.e. such that

$$[\alpha_n\beta_n] = [g_{j_{n+1},j_n}] \quad \text{and} \quad [\beta_{n+1}\alpha_n] = [f_{i_{n+1},i_n}]$$

as elements of $\mathrm{KK}(B_{j_n}, B_{j_{n+1}})$ respectively $\mathrm{KK}(A_{i_n}, A_{i_{n+1}})$.

Observation 3.11 Consider two KK-shape equivalent systems (A_n, f_n) and (B_n, g_n). By associativity of the Kasparov product, the diagram

$$\begin{array}{ccccccc}
\mathrm{K}_*(A_1) & \xrightarrow{\cdot[f_1]} & \mathrm{K}_*(A_2) & \xrightarrow{\cdot[f_2]} & \mathrm{K}_*(A_3) & \xrightarrow{\cdot[f_3]} & \cdots \\
{\scriptstyle \cdot[\beta_1]}\uparrow & \searrow^{\cdot[\alpha_1]} & {\scriptstyle \cdot[\beta_2]}\uparrow & \searrow^{\cdot[\alpha_2]} & {\scriptstyle \cdot[\beta_3]}\uparrow & \searrow^{\cdot[\alpha_3]} & \\
\mathrm{K}_*(B_1) & \xrightarrow{\cdot[g_1]} & \mathrm{K}_*(B_2) & \xrightarrow{\cdot[g_2]} & \mathrm{K}_*(B_3) & \xrightarrow{\cdot[g_3]} & \cdots
\end{array}$$

commutes. One now sees, as the maps $\cdot[\alpha_n]$ and $\cdot[\beta_n]$ are positive, that the maps induce positive maps, $\varinjlim \mathrm{K}_*(A_n) \to \varinjlim \mathrm{K}_*(B_n)$ given by $a[f_{\infty,n}] \mapsto a[\alpha_n][g_{\infty,n+1}]$ and $\varinjlim \cdot[\beta_n]\colon \varinjlim \mathrm{K}_*(B_n) \to \varinjlim \mathrm{K}_*(A_n)$ given by $b[g_{\infty,n}] \mapsto b[\beta_n][f_{\infty,n}]$, that are each other's inverses. As $\mathrm{K}_*(\varinjlim A_n) = \varinjlim \mathrm{K}_*(A_n)$ and $\mathrm{K}_*(\varinjlim B_n) = \varinjlim \mathrm{K}_*(B_n)$, we conclude that $\varinjlim A_n$ and $\varinjlim B_n$ have the same ordered K-theory.

As we will construct isomorphisms between AD algebras via KK-shape equivalence of inductive systems of direct sums of matrix algebras over building blocks, the following theorem is crucial to us. See [Ell93, 7.1] or [DG97, 7.3] for a proof.

Theorem 3.12 Consider real rank zero AD algebras A and B that are given as the inductive limits

$$A = \varinjlim (A_i, f_i) \quad \text{and} \quad B = \varinjlim (B_i, g_i)$$

of systems of direct sums of matrix algebras over building blocks. Assume that the inductive systems (A_i, f_i) and (B_i, g_i) are KK-shape equivalent, and let $(A_{i_n}, f_{i_{n+1}, i_n})$ and $(B_{j_n}, g_{j_{n+1}, j_n})$ denote subsystems giving a KK-shape equivalent and $\alpha_n\colon A_{i_n} \to B_{j_{n+1}}$ and $\beta_n\colon B_{j_n} \to A_{i_n}$ the corresponding $*$-homomorphisms satisfying

$$[\alpha_n \beta_n] = [g_{j_{n+1}, j_n}] \quad \text{and} \quad [\beta_{n+1} \alpha_n] = [f_{i_{n+1}, i_n}].$$

Then the limits A and B are isomorphic, and an isomorphism $\alpha\colon A \to B$ may be chosen to be compatible with the given KK-shape equivalence, i.e. such that for each n

$$[\alpha f_{\infty, i_n}] = [g_{\infty, j_{n+1}} \alpha_n] \quad \text{and} \quad [\alpha^{-1} g_{\infty, j_n}] = [f_{\infty, i_n} \beta_n]$$

as elements of $\mathrm{KK}(A_{i_k}, B)$ respectively $\mathrm{KK}(B_{j_k}, A)$.

4 K-theory with coefficients

We are now ready to define K-theory with coefficients, its associated natural transformations, and finally our invariant. In this section we will also determine the image of our invariant on the building blocks $C(S^1)$ and $\mathbb{I}_n^\sim$. The constructions, and most of the proofs, are from [Eil95], with a few details added or changed.

For any nuclear, separable C^*-algebra one can define K-theory with coefficients as well as its associated natural transformations. In the following we will however only be dealing with AD algebras, as this allows us to use some shortcuts.

4.1 The ordered $(\mathbb{Z}/2)^2$-graded group $\mathrm{K}_*(-;\mathbb{Z}\oplus\mathbb{Z}/n)$

Firstly, we define K-theory with coefficients. As noted on p. 6, $(\mathrm{KK}(\mathbb{I}_n^\sim, A), \mathrm{KK}(\mathbb{I}_n^\sim, A)^+)$ is an ordered group, while $(\mathrm{KK}(\mathbb{I}_n, A), \mathrm{KK}(\mathbb{I}_n, A)^+)$ is not.

Definition 4.1 For an AD algebra A we define for all $n \in \mathbb{N}\backslash\{1\}$ ordered K-theory with coefficients in $\mathbb{Z}/n$ as

- $\mathrm{K}_0(A;\mathbb{Z}/n) = \mathrm{KK}(\mathbb{I}_n, A)$;
- $\mathrm{K}_1(A;\mathbb{Z}/n) = \mathrm{KK}(\mathbb{I}_n, \mathrm{S}\,A)$;
- $\mathrm{K}_0(A;\mathbb{Z}\oplus\mathbb{Z}/n) = \mathrm{KK}(\mathbb{I}_n^\sim, A)$ with positive cone $\mathrm{K}_0(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+ = \mathrm{KK}(\mathbb{I}_n^\sim, A)^+$.

Lemma 4.2 The groups $\mathrm{K}_0(A;\mathbb{Z}/n)$ and $\mathrm{K}_1(A;\mathbb{Z}/n)$ are $\mathbb{Z}/n$-modules for any AD algebra A and any n.

Proof. Let $x \in \mathrm{KK}(\mathbb{I}_n, A)$ be given. As $[\mathrm{id}_{\mathbb{I}_n}]$ is a left unit for $\mathrm{KK}(\mathbb{I}_n, A)$, we get that $nx = (n[\mathrm{id}_{\mathbb{I}_n}])x$. And according to 3.1, $\mathrm{diag}(\mathrm{id}_{\mathbb{I}_n}, \dots, \mathrm{id}_{\mathbb{I}_n})\colon \mathbb{I}_n \to M_n(\mathbb{I}_n)$ is null homotopic, hence $n[\mathrm{id}_{\mathbb{I}_n}] = [\mathrm{diag}(\mathrm{id}_{\mathbb{I}_n}, \dots, \mathrm{id}_{\mathbb{I}_n})] = 0$. And therefore $nx = 0$, as desired. $n\,\mathrm{K}_1(A;\mathbb{Z}/n) = 0$ can be proved in the same way. ♡

Of course we wish for $\mathrm{K}_0(A;\mathbb{Z}\oplus\mathbb{Z}/n)$ to be the direct sum of $\mathrm{K}_0(A)$ and $\mathrm{K}_0(A;\mathbb{Z}/n)$.

Observation 4.3 For any AD algebra A, we can consider the split-exact sequences

$$0 \longrightarrow \mathrm{S}\,A \underset{}{\overset{\iota}{\rightleftarrows}} A \otimes C(S^1) \underset{\mathrm{id}\otimes\eta}{\overset{\mathrm{id}\otimes\delta}{\rightleftarrows}} A \longrightarrow 0\,,$$

where $\delta\colon C(S^1) \to \mathbb{C}$ is defined as $\delta(f) = f(t)$ for some fixed $t \in S^1$, and

$$0 \longrightarrow \mathbb{I}_n \overset{\iota}{\rightleftarrows} \mathbb{I}_n^\sim \underset{\eta}{\overset{\delta_0}{\rightleftarrows}} \mathbb{C} \longrightarrow 0\,.$$

Since A is separable and nuclear, lemma 2.8 ensures us that these sequences induce a KK-natural diagram with split-exact rows and columns and with all its twenty-four maps being positive:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{KK}(\mathbb{C}, A) & \underset{[\eta]\cdot}{\overset{[\delta_0]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n^\sim, A) & \underset{}{\overset{[\iota]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n, A) & \longrightarrow & 0 \\
 & & {\cdot[\mathrm{id}\otimes\delta]} \uparrow \downarrow {\cdot[\mathrm{id}\otimes\eta]} & & {\cdot[\mathrm{id}\otimes\delta]} \uparrow \downarrow {\cdot[\mathrm{id}\otimes\eta]} & & {\cdot[\mathrm{id}\otimes\delta]} \uparrow \downarrow {\cdot[\mathrm{id}\otimes\eta]} & & \\
0 & \longrightarrow & \mathrm{KK}(\mathbb{C}, A \otimes C(S^1)) & \underset{[\eta]\cdot}{\overset{[\delta_0]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n^\sim, A \otimes C(S^1)) & \underset{}{\overset{[\iota]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n, A \otimes C(S^1)) & \longrightarrow & 0 \\
 & & {\cdot[\iota]} \uparrow \downarrow & & {\cdot[\iota]} \uparrow \downarrow & & {\cdot[\iota]} \uparrow \downarrow & & \\
0 & \longrightarrow & \mathrm{KK}(\mathbb{C}, \mathrm{S}\, A) & \underset{[\eta]\cdot}{\overset{[\delta_0]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n^\sim, \mathrm{S}\, A) & \underset{}{\overset{[\iota]\cdot}{\rightleftarrows}} & \mathrm{KK}(\mathbb{I}_n, \mathrm{S}\, A) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

It follows from the associativity of the Kasparov product that the four solid squares and the four dotted squares are commutative.

Hence the diagram above defines a $(\mathbb{Z}/2)^2$-grading of $\mathrm{KK}(\mathbb{I}_n^\sim, A \otimes C(S^1))$ where the positive cone of $\mathrm{KK}(\mathbb{C}, A)$ corresponds to the positive cone inherited from $\mathrm{KK}(\mathbb{I}_n^\sim, A)^+$ or $\mathrm{KK}(\mathbb{C}, A \otimes C(S^1))^+$ and where these positive cones correspond to those inherited from $\mathrm{KK}(\mathbb{I}_n^\sim, A \otimes C(S^1))^+$.

In light of observation 4.3, we can now make the following definition. Notice how the order on $\mathrm{K}_*(A)$ restricts to the order on $\mathrm{K}_0(A)$, by 4.3.

Definition 4.4 For an *AD* algebra A we define the positive cone in $\mathrm{K}_*(A)$ as $\mathrm{K}_*(A)^+ = \mathrm{KK}(\mathbb{C}, A \otimes C(S^1))^+$.

Remark 4.5 One usually defines the positive cone in $\mathrm{K}_*(A)$ as those couples $([p]_0, [u]_1)$ where $p \in M_n(A)$ is a projection and the unitary $u \in M_n(A)$ lies in $pM_n(A)p$. According to [Bla06, V.2.4.31], this standard order on $\mathrm{K}_*(A)$ agrees with the K_0-order on $\mathrm{K}_0(A \otimes C(S^1))$. Notice that in the case where the C^*-algebra A is simple, any (x, y) in $\mathrm{K}_*(A)$ with $x > 0$, is positive. The same applies to the special case where $[1]_0$ generates $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$ has a set of generators that lie in A.

Lemma 4.6 For any *AD* algebra A, $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ and $(\mathrm{K}_*(A), \mathrm{K}_*(A)^+)$ are ordered groups.

Proof. When we make the identification $\mathrm{K}_*(A) = \mathrm{K}_0(A \otimes C(S^1))$, we see that by the continuity of $\mathrm{K}_0(-)$ stated in 2.12, it suffices to proof that $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is an ordered group when A is either a finite sum of matrices over building blocks or such a sum tensored by $C(S^1)$.

But as mentioned, $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is an ordered group when A is stably finite, according to [Bla06, V.2.4.8], and as we have seen, the building blocks (and therefore also finite direct sums of matrices over them) are stably finite as they are of stable rank one.

Hence, we are completely done when we have shown that $A \otimes C(S^1)$ is stably finite when A is a finite direct sum of matrices over building blocks. That is, we must show that $M_n(A \otimes C(S^1))$ is finite for any n, and as $M_n(A \otimes C(S^1)) = M_n(A) \otimes C(S^1)$, it suffices to prove that $A \otimes C(S^1)$ is finite.

So let $p \in C(S^1, A)$ be a projection and assume that p is Murray-von Neumann equivalent to 1. Then we have a partial isometry $v \in C(S^1, A)$ such that $vv^* = p$ and $v^*v = 1$. But then this also holds point-wise, hence for any $x \in S^1$, $p(x)$ is Murray-von Neumann equivalent to 1 in A, so as A is finite, $p(x) = 1$ for any $x \in S^1$, hence $p = 1$ and we conclude that $C(S^1, A)$ is finite. ♡

Let us determine the K-groups of our building blocks, and let us also fix some generators for the groups as this will allow us to describe maps between them by matrix representations.

To describe the ordered K-groups of our building blocks, we will need to establish some notation. By $\mathbb{Z} \oplus_{\geq} (\mathbb{Z}/n_1 \oplus \cdots \mathbb{Z}/n_k)$ we denote the group $\mathbb{Z} \oplus \mathbb{Z}/n_1 \oplus \cdots \mathbb{Z}/n_k$ equipped with the order that an element $(x, x_1, \ldots, x_k)$ is said to be positive if we have for some representatives $x_1, \ldots, x_k$ that $x \geq x_i \geq 0$ for all i. By $\mathcal{Z}(m; n)$ we denote the subgroup $\{(x, y) \in \mathbb{Z}/n \oplus \mathbb{Z}/n \mid x \equiv y \pmod{\frac{n}{(n,m)}}\}$, and $\mathbb{Z} \oplus_{\geq} \mathcal{Z}(m; n)$ is then equipped with the order it inherits as a subgroup of $\mathbb{Z} \oplus_{\geq} (\mathbb{Z}/n \oplus \mathbb{Z}/n)$. Given abelian groups G and H where G is an ordered group, we denote by $G \oplus_{\vdash} H$ the group $G \oplus H$ equipped with *the strict order arrising from* G, i.e. the order that $(g, h) \geq 0$ exactly when $(g, h) = (0, 0)$ or $g > 0$.

To fix generators for the groups $\mathrm{K}_1(C(S^1))$ and $\mathrm{K}_1(\mathbb{I}_m^\sim)$, we define the two unitaries $u \in C(S^1)$ and $u_m \in \mathbb{I}_m^\sim$ as $u(t) = t$ and $u_m(t) = \operatorname{diag}(e^{2\pi i t}, 1, \ldots, 1)$. The following is well-known, and we therefore keep the proof brief; the details about $\mathbb{I}_m$ and $\mathbb{I}_m^\sim$ are from [RLL00, p. 220].

Lemma 4.7 The ordered K_*-groups are $\mathrm{K}_*(\mathbb{C}) = \mathbb{Z} \oplus 0$, $\mathrm{K}_*(C(S^1)) = \mathbb{Z} \oplus_{\vdash} \mathbb{Z}$ and $\mathrm{K}_*(\mathbb{I}_m^\sim) = \mathbb{Z} \oplus_{\vdash} \mathbb{Z}/m$, where $0 \oplus \mathrm{K}_1(C(S^1))$ is generated by $[u]_1 = (0, 1)$ and $0 \oplus \mathrm{K}_1(\mathbb{I}_m^\sim)$ is generated by $(0, 1) = [u_m]_1$, and furthermore $u_m \sim_1 \operatorname{diag}(1, \ldots, 1, e^{2\pi i t}, 1, \ldots, 1)$.

Proof. Two projections $p, q \in M_n$ are Murray-von Neumann equivalent if and only if they have the same trace, hence $\mathrm{K}_0(\mathbb{C})^+ = \mathbb{N}_0$, and $\mathrm{K}_0(\mathbb{C})$ is therefore isomorphic to $\mathbb{Z}$ equipped with the usual order. Also, as the group of unitary $n \times n$ matrices is path-connected, $\mathrm{K}_1(\mathbb{C}) = 0$.

As $C(S^1) = C_0((0, 1))^\sim$,

$$\mathrm{K}_0(C(S^1)) = \mathrm{K}_0(C_0((0, 1))) \oplus \mathbb{Z} = \mathrm{K}_1(\mathbb{C}) \oplus \mathbb{Z} = \mathbb{Z},$$

and likewise $\mathrm{K}_1(C(S^1)) = \mathrm{K}_1(C_0((0, 1))) = \mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$. As the projection $1 \in \mathbb{C}$ generates $\mathrm{K}_0(\mathbb{C})$, we see via the Bott map (2.2) that $\beta_{\mathbb{C}}([1]_0) = [f_1]_1$ generates $\mathrm{K}_1(\mathrm{S}\,\mathbb{C}) = \mathrm{K}_1(C(S^1))$, so as $f_1(t) = e^{2\pi i t}$ corresponds to u via the identification of the one-point compactification of $(0, 1)$ with S^1, $[u]_1$ generates $\mathrm{K}_1(C(S^1))$.

Considering the extension

$$0 \longrightarrow \mathrm{S}\, M_m \xrightarrow{\iota} \mathbb{I}_m \xrightarrow{\delta_1} \mathbb{C} \longrightarrow 0$$

and noticing that it is isomorphic to the mapping cone sequence of the map $\eta\colon \mathbb{C} \to M_m$, cf. 2.5, we get, by 2.6, a six-term exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{K}_0(\mathbb{I}_m) & \xrightarrow{\mathrm{K}_0(\delta_1)} & \mathrm{K}_0(\mathbb{C}) \\ \uparrow & & & & \downarrow {\scriptstyle \mathrm{K}_0(\eta)} \\ & & & & \mathrm{K}_0(M_m) \\ & & & & \| \\ 0 & \longleftarrow & \mathrm{K}_1(\mathbb{I}_m) & \xleftarrow[\mathrm{K}_1(\iota)]{} & \mathrm{K}_1(\mathrm{S}\, M_m) \end{array}$$

as $\mathrm{K}_1(\mathbb{C}) = 0$ and $\mathrm{K}_0(\mathrm{S}\, M_m) = 0$. Now, $\mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{S}\, M_m) = \mathbb{Z}$, and as $\mathrm{K}_0(\eta)([1]_0) = [\mathrm{diag}(1, \dots, 1)]_0 = m[1]_0$, we see that $\mathrm{K}_0(\mathbb{I}_m) = 0$ and $\mathrm{K}_1(\mathbb{I}_m) = \mathbb{Z}/m$. Hence, $\mathrm{K}_*(\mathbb{I}_m^\sim) = \mathbb{Z} \oplus \mathbb{Z}/m$ as groups.

By surjectivity of $\mathrm{K}_1(\iota)$, we only need to show that u_m generates $\mathrm{K}_1(\mathrm{S}\, M_m)$ to conclude that it generates $\mathrm{K}_1(\mathbb{I}_m) = \mathrm{K}_1(\mathbb{I}_m^\sim)$. And as the Bott isomorphism $\beta_{M_m}\colon \mathrm{K}_0(M_m) \to \mathrm{K}_1(\mathrm{S}\, M_m)$ maps the genators $p = \mathrm{diag}(1, 0, \dots, 0) \in M_m$ of $\mathrm{K}_0(M_m)$ to $[f_p]_1$ with $f_p(t) = e^{2\pi i t} p + (1_n - p) = u_n(t)$, $[u_n]_1$ generates $\mathrm{K}_1(\mathrm{S}\, M_n)$.

By 4.5, we see that $\mathrm{K}_*(C(S^1))$ and $\mathrm{K}_*(\mathbb{I}_m^\sim)$ have the strict order, as desired.

As for the claim that

$$u_1 \sim_1 \mathrm{diag}(1, \dots, 1, e^{2\pi i t}, 1, \dots, 1) \in \mathbb{I}_m^\sim,$$

one sees it by letting $v \in M_m$ denote a unitary satisfying

$$v\, \mathrm{diag}(1, 0, \dots, 0) v^* = \mathrm{diag}(0, \dots, 0, 1, 0 \dots, 0)$$

and letting $s \mapsto v_s$ denote a continuous path of unitaries in M_m with $v_0 = 1_m$ and $v_1 = v$, and then noting that $s \mapsto v_s u_m v_s^*$ is a continuous path of unitaries in $\mathbb{I}_m^\sim$ from u_m to $\mathrm{diag}(1, \dots, 1, e^{2\pi i t}, 1, \dots, 1)$. ♡

As we will be using the Universal Coefficient Theorem (2.11), we need the following lemma.

Lemma 4.8 For any abelian group G, $\mathrm{Ext}^1(\mathbb{Z}/n, G) = G/nG$.

Proof. Clearly,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n$, hence $\mathrm{Ext}^1(\mathbb{Z}/n, G)$ is the 1^{st} cohomology of the complex

$$0 \longleftarrow \mathrm{Hom}(\mathbb{Z}, G) \xleftarrow{n} \mathrm{Hom}(\mathbb{Z}, G) \longleftarrow 0\,.$$

As $\mathrm{Hom}(\mathbb{Z}, G) = G$, this is G/nG, as desired. ♡

Lemma 4.9 The ordered group $\mathrm{K}_0(\mathbb{C};\mathbb{Z}\oplus\mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}\oplus_{\geq}\mathbb{Z}/n$ and is generated by $(1,0)=[\delta_0]$ and $(1,1)=[\delta_1]$, the ordered group $\mathrm{K}_0(C(S^1);\mathbb{Z}\oplus\mathbb{Z}/n)$ isomorphic to $\mathbb{Z}\oplus_{\geq}\mathbb{Z}\oplus\mathbb{Z}/n$ and generated by $(1,0)=[\eta\delta_0]$ and $(1,1)=[\eta\delta_1]$, and the ordered group $\mathrm{K}_0(\mathbb{I}_m^{\sim};\mathbb{Z}\oplus\mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}\oplus_{\geq}\mathcal{Z}(m;n)$ and is generated by the four elements $(1,0,0)=[\eta\delta_0]$, $(1,1,1)=[\eta\delta_1]$, $\frac{n}{(n,m)}(1,0,1)=[\mathrm{in}_{m,n}]$ and $\frac{n}{(n,m)}(1,1,0)=[\overline{\mathrm{in}}_{m,n}]$.

Proof. In 4.7 we saw that $\mathrm{K}_*(\mathbb{I}_n)=0\oplus\mathbb{Z}/n$, so by the UCT (2.11) we see that

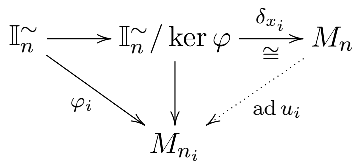
$$\begin{aligned}\mathrm{KK}(\mathbb{I}_n,A)&=\mathrm{Ext}^1(0,\mathrm{K}_1(A))\oplus\mathrm{Ext}^1(\mathbb{Z}/n,\mathrm{K}_0(A))\oplus\mathrm{Hom}(0,\mathrm{K}_0(A))\oplus\mathrm{Hom}(\mathbb{Z}/n,\mathrm{K}_1(A))\\&=\mathrm{K}_0(A)/n\oplus\mathrm{K}_1(A)[n],\end{aligned}$$

hence $\mathrm{K}_0(\mathbb{C};\mathbb{Z}/n)=\mathbb{Z}/n$, $\mathrm{K}_0(C(S^1);\mathbb{Z}/n)=\mathbb{Z}/n$ and $\mathrm{K}_0(\mathbb{I}_m^{\sim};\mathbb{Z}/n)=\mathbb{Z}/n\oplus\mathbb{Z}/(n,m)$ by 4.7. This tells us nothing about the order on $\mathrm{K}_0(-;\mathbb{Z}\oplus\mathbb{Z}/n)$, however.

We desire to define a map

$$\Lambda_{\mathbb{C}}\colon\ \mathrm{KK}(\mathbb{I}_n^{\sim},\mathbb{C})\to\mathbb{Z}\oplus\mathbb{Z}/n,$$

and as $\mathrm{KK}(\mathbb{I}_n^{\sim},\mathbb{C})$ is an ordered group, it suffices to define $\Lambda_{\mathbb{C}}$ on $\mathrm{KK}(\mathbb{I}_n^{\sim},\mathbb{C})^+$. Consider $\varphi\colon\mathbb{I}_n^{\sim}\to M_k$. As φ is a finite-dimensional representation of $\mathbb{I}_n^{\sim}$, it is the direct sum $\varphi=\mathrm{diag}(\varphi_1,\dots,\varphi_r)$ of irreducible representations of $\mathbb{I}_n^{\sim}$. Now, as the kernel of $\varphi_i\colon\mathbb{I}_n^{\sim}\to M_{n_i}$ is a maximal ideal in $\mathbb{I}_n^{\sim}$, $\ker\varphi_i=\{f\in\mathbb{I}_n^{\sim}\mid f(x_i)=0\}$ for some $x_i\in[0,1]$, and δ_{x_i} induces an isomorphism from $\mathbb{I}_n^{\sim}/\ker\varphi_i$ to M_n, whereby φ_i induces an irreducible representation from M_n to M_{n_i}, which must be on the form $x\mapsto u_i^*xu_i$ for some unitary u in M_n and with $n_i=n$,



so $\varphi_i=\mathrm{ad}(u_i)\delta_{x_i}$. Ergo $\varphi=\mathrm{ad}(u)\,\mathrm{diag}(\delta_{x_1},\dots,\delta_{x_r})$ with $u=\mathrm{diag}(u_1,\dots,u_r)$ where $u_i=1$ when $x_i\in\{0,1\}$. For $i\in\{0,1\}$, let c_i denote the number of times δ_i accours in $\mathrm{diag}(\delta_{x_1},\dots,\delta_{x_r})$, and let $d=r-c_0-c_1$. Consider another $*$-homomorphism $\tilde{\varphi}\colon\mathbb{I}_n^{\sim}\to M_k$ homotopic to φ, and do the same, ending with integers $\tilde{c}_0$, $\tilde{c}_1$ and $\tilde{d}$. Now, δ_t is homotopic to $\mathrm{diag}(\delta_i,\dots,\delta_i)\colon\mathbb{I}_n^{\sim}\to M_n$ for all $t\in(0,1)$ and $i\in\{0,1\}$, but δ_0 and δ_1 are not homotopic, and $\mathrm{diag}(\delta_i,\dots,\delta_i,0,\dots,0)\colon\mathbb{I}_n^{\sim}\to M_n$ is not homotopic to δ_t, hence

$$c_0+c_1+nd=\tilde{c}_0+\tilde{c}_1+n\tilde{d}\quad\text{and}\quad c_1\equiv\tilde{c}_1\pmod n$$

as $\delta_0^{\oplus c_0}\oplus\delta_1^{\oplus c_1}\oplus\delta_{\frac12}^{\oplus d}$ and $\delta_0^{\oplus\tilde{c}_0}\oplus\delta_1^{\oplus\tilde{c}_1}\oplus\delta_{\frac12}^{\oplus\tilde{d}}$ are homotopic. This means that we can define $\Lambda_{\mathbb{C}}$ on $\mathrm{KK}(\mathbb{I}_n^{\sim},\mathbb{C})^+$ by

$$[\varphi]\mapsto(c_0+c_1+nd,c_1)\in\mathbb{Z}\oplus\mathbb{Z}/n.$$

By construction the map is additive, hence we get a group homomorphism $\mathrm{KK}(\mathbb{I}_n^{\sim})\to\mathbb{Z}\oplus\mathbb{Z}/n$.

As $(1,i) = \Lambda_{\mathbb{C}}([\delta_i]) \in \operatorname{im}\Lambda_{\mathbb{C}}$, $\Lambda_{\mathbb{C}}$ is surjective. Since $\mathrm{KK}(\mathbb{I}_n,\mathbb{C}) = \mathbb{Z}/n$ is a torsion group, $\Lambda_{\mathbb{C}}$ maps this part of $\mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})$ to $0\oplus\mathbb{Z}/n$, and as $\Lambda_{\mathbb{C}}([\varphi\delta_0]) \in \mathbb{Z}\oplus 0$, we conclude that $\Lambda_{\mathbb{C}}$ is graded, cf. 4.3. As $\Lambda_{\mathbb{C}}$ is a graded, surjective group homomorphism from $\mathbb{Z}\oplus\mathbb{Z}/n$ to $\mathbb{Z}\oplus\mathbb{Z}/n$, we conclude that it is injective, as any epimorphism $\mathbb{Z}\to\mathbb{Z}$ and any epimorphism $\mathbb{Z}/n\to\mathbb{Z}/n$ is.

Clearly,

$$\Lambda_{\mathbb{C}}(\mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})^+) = (\mathbb{Z}\oplus_{\geq}\mathbb{Z}/n)^+,$$

hence $\Lambda_{\mathbb{C}}$ is an graded order-isomorphism from $\mathrm{KK}(\mathbb{I}_n^\sim)^+$ to $\mathbb{Z}\oplus_{\geq}\mathbb{Z}/n$, as desired. Furthermore, as $\Lambda_{\mathbb{C}}([\delta_0])$ and $\Lambda_{\mathbb{C}}([\delta_1])$ generate $\mathbb{Z}\oplus\mathbb{Z}/n$, $[\delta_0]$ and $[\delta_1]$ generate $\mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})$.

Now, define

$$\Lambda_{C(S^1)}\colon \mathrm{KK}(\mathbb{I}_n^\sim, C(S^1)),\quad \Lambda_{C(S^1)} = \Lambda_{\mathbb{C}}\circ(\cdot[\delta])$$

where $\delta\colon C(S^1)\to\mathbb{C}$ is given by $\delta(f) = f(1)$. As $[\eta][\delta] = [\mathrm{id}_{\mathbb{C}}]$, $\cdot[\delta]$ and therefore also $\Lambda_{C(S^1)}$ is onto. And by associativity of the Kasparov product, $\cdot[\delta]\colon \mathrm{KK}(\mathbb{I}_n^\sim, C(S^1)) \to \mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})$ is graded, cf. 4.3. Hence $\Lambda_{\mathbb{C}}$ is graded. Again, any graded, surjective homomorphism from $\mathbb{Z}\oplus\mathbb{Z}/n$ to $\mathbb{Z}\oplus\mathbb{Z}/n$ must be injective, ergo $\Lambda_{C(S^1)}$ is also injective. As $[\eta][\delta] = [\mathrm{id}_{\mathbb{C}}]$, we see that $\mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})^+ = \{x[\delta] \mid x\in\mathrm{KK}(\mathbb{I}_n^\sim, C(S^1))\}$, hence

$$\Lambda_{C(S^1)}(\mathrm{KK}(\mathbb{I}_n^\sim, C(S^1))^+) = \Lambda_{\mathbb{C}}(\mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{C})^+)$$

whereby we conclude that $\Lambda_{C(S^1)}$ is a graded order-isomorphism from $\mathrm{KK}(\mathbb{I}_n^\sim, C(S^1))$ to $\mathbb{Z}\oplus_{\geq}\mathbb{Z}/n$. And as $\Lambda_{C(S^1)}([\eta\delta_i]) = (1,i)$, since $\delta\eta = \mathrm{id}_{\mathbb{C}}$, we see that $\mathrm{KK}(\mathbb{I}_n^\sim, C(S^1))$ is generated by $[\eta\delta_0]$ and $[\eta\delta_1]$.

Finally, define

$$\Lambda_{\mathbb{I}_m^\sim}\colon \mathrm{KK}(\mathbb{I}_n^\sim,\mathbb{I}_m^\sim)\to(\mathbb{Z}\oplus\mathbb{Z}/n)^2,\quad \Lambda_{\mathbb{I}_m^\sim} = (\Lambda_{\mathbb{C}}\circ(\cdot[\delta_0]))\oplus(\Lambda_{\mathbb{C}}\circ(\cdot[\delta_1])).$$

Again, $\Lambda_{\mathbb{I}_n^\sim}$ is graded. But it is not surjective. Since $\delta_0\eta = \delta_1\eta$, elements from $\mathrm{KK}(\mathbb{C},\mathbb{I}_m^\sim)$ are mapped to elements of the form $(x,0,x,0)$, cf. 4.3. Now, by composition of maps we see that

$$\begin{aligned}
\Lambda_{\mathbb{I}_m^\sim}([\eta\delta_i]) &= (\Lambda_{\mathbb{C}}([\delta_i]),\Lambda_{\mathbb{C}}([\delta_i])) = (1,i,1,i)\\
\Lambda_{\mathbb{I}_m^\sim}([\mathrm{in}_{m,n}]) &= (\Lambda_{\mathbb{C}}(\frac{n}{(n,m)}[\delta_0]),\Lambda_{\mathbb{C}}(\frac{n}{(n,m)}[\delta_1])) = \frac{n}{(n,m)}(1,0,1,1)\\
\Lambda_{\mathbb{I}_m^\sim}([\overline{\mathrm{in}}_{m,n}]) &= (\Lambda_{\mathbb{C}}(\frac{n}{(n,m)}[\delta_1]),\Lambda_{\mathbb{C}}(\frac{n}{(n,m)}[\delta_0])) = \frac{n}{(n,m)}(1,1,1,0),
\end{aligned}$$

whereby we conclude that

$$\{(x,a,x,b)\mid a\equiv b \pmod{\frac{n}{(n,m)}}, x\in\mathbb{Z}\}\subseteq\operatorname{im}\Lambda_{\mathbb{I}_m^\sim}.$$

By this, we see that $\Lambda_{\mathbb{I}_m^\sim}$ gives a epimorphism from $\mathrm{KK}(\mathbb{C},\mathbb{I}_m^\sim)$ to the diagonal in $\mathbb{Z}\oplus\mathbb{Z}$, so as $\mathrm{KK}(\mathbb{C},\mathbb{I}_m^\sim) = \mathbb{Z}$, we conclude that $\Lambda_{\mathbb{I}_m^\sim}$ is injective on $\mathrm{KK}(\mathbb{C},\mathbb{I}_m^\sim)$. Also, as $\{(a,b)\in\mathbb{Z}/n\oplus\mathbb{Z}/n \mid a\equiv b \pmod{\frac{n}{(n,m)}}\}$ has $n(n,m)$ elements and therefore as many elements as $\mathrm{KK}(\mathbb{I}_n,\mathbb{I}_m^\sim)$ has, we conclude that $\Lambda_{\mathbb{I}_m^\sim}$ is injective on $\mathrm{KK}(\mathbb{I}_n,\mathbb{I}_m^\sim)$ and that

$\{(x,a,x,b) \mid a \equiv b \pmod{\frac{n}{(n,m)}}, x \in \mathbb{Z}\}$ is the entire image of $\Lambda_{\mathbb{I}^{\sim}_{m}}$. So as $\Lambda_{\mathbb{I}^{\sim}_{m}}$ is injective, and as $\Lambda_{\mathbb{I}^{\sim}_{m}}([\eta\delta_0])$, $\Lambda_{\mathbb{I}^{\sim}_{m}}([\eta\delta_1])$, $\Lambda_{\mathbb{I}^{\sim}_{m}}([\mathrm{in}_{m,n}])$ and $\Lambda_{\mathbb{I}^{\sim}_{m}}([\overline{\mathrm{in}}_{m,n}])$ generate $\operatorname{im}\Lambda_{\mathbb{I}^{\sim}_{m}}$, the four elements $[\eta\delta_0]$, $[\eta\delta_1]$, $[\mathrm{in}_{m,n}]$ and $[\overline{\mathrm{in}}_{m,n}]$ generate $\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{I}^{\sim}_{m})$.

As for the positive cone, we see that

$$\begin{aligned} \Lambda_{\mathbb{I}^{\sim}_{m}}(\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{I}^{\sim}_{m})^+) &\supseteq \operatorname{span}_{\mathbb{N}_0}\{\Lambda_{\mathbb{I}^{\sim}_{m}}([\eta\delta_0]), \Lambda_{\mathbb{I}^{\sim}_{m}}([\eta\delta_1]), \Lambda_{\mathbb{I}^{\sim}_{m}}([\mathrm{in}_{m,n}]), \Lambda_{\mathbb{I}^{\sim}_{m}}([\overline{\mathrm{in}}_{m,n}])\} \\ &= \{(x,a,x,b) \in \operatorname{im}\Lambda_{\mathbb{I}^{\sim}_{m}} \mid x \geq a \geq 0, x \geq b \geq 0\}, \end{aligned}$$

while at the same time

$$\begin{aligned} \Lambda_{\mathbb{I}^{\sim}_{m}}(\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{I}^{\sim}_{m})^+) &\subseteq (\Lambda_{\mathbb{C}}(\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{C})^+) \oplus \Lambda_{\mathbb{C}}(\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{C})^+)) \cap \operatorname{im}\Lambda_{\mathbb{I}^{\sim}_{m}} \\ &= \{(x,a,y,b) \in (\mathbb{Z} \oplus \mathbb{Z}/n)^2 \mid x \geq a \geq 0, y \geq b \geq 0\} \cap \operatorname{im}\Lambda_{\mathbb{I}^{\sim}_{m}} \\ &= \{(x,a,x,b) \in \operatorname{im}\Lambda_{\mathbb{I}^{\sim}_{m}} \mid x \geq a \geq 0, x \geq b \geq 0\}, \end{aligned}$$

as the maps $\cdot[\delta_0]$ and $\cdot[\delta_1]$ are positive. So, via $(x,a,x,b) \mapsto (x,a,b)$, we get a graded order-isomorphism from $\mathrm{KK}(\mathbb{I}^{\sim}_{n}, \mathbb{I}^{\sim}_{m})$ to $\mathbb{Z} \oplus_{\geq} \mathcal{Z}(m;n)$. ♡

4.2 The natural transformations ρ_i^n and β_i^n

Definition 4.10 Given an *AD* algebra A we define for $i \in \{0,1\}$ *the reduction maps* $\rho_n^i\colon \mathrm{K}_i(A) \to \mathrm{K}_i(A;\mathbb{Z}/n)$ as the Kasparov multiplication $x \mapsto [\delta_1]x$ by $[\delta_1] \in \mathrm{KK}(\mathbb{I}_n, \mathbb{C})$. And we define *the Bockstein maps* $\beta_n^i\colon \mathrm{K}_i(A;\mathbb{Z}/n) \to \mathrm{K}_{i+1}(A)$ as the Kasparov multiplication $x \mapsto [\iota]x$ by $[\iota] \in \mathrm{KK}(\mathrm{S}\,M_n, \mathbb{I}_n)$.

Lemma 4.11 For any *AD* algebra A we obtain a KK-natural six-term exact sequence of the form

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{\rho_n^0} & \mathrm{K}_0(A;\mathbb{Z}/n) & \xrightarrow{\beta_n^0} & \mathrm{K}_1(A) \\ {\scriptstyle n}\uparrow & & & & \downarrow{\scriptstyle n} \\ \mathrm{K}_0(A) & \xleftarrow[\beta_n^1]{} & \mathrm{K}_1(A;\mathbb{Z}/n) & \xleftarrow[\rho_n^1]{} & \mathrm{K}_1(A). \end{array}$$

Proof. Consider again the short-exact sequence

$$0 \longrightarrow \mathrm{S}\,M_n \xrightarrow{\iota} \mathbb{I}_n \xrightarrow{\delta_1} \mathbb{C} \longrightarrow 0$$

and notice that it is isomorphic to the mapping cone sequence associated to the mapping $\eta\colon \mathbb{C} \to M_n$, cf. 2.5. According to 2.6 the six-term exact sequence associated to the sequence above is of the form

$$\begin{array}{ccccc} \mathrm{KK}(\mathbb{C}, A) & \xrightarrow{[\delta_1]\cdot} & \mathrm{KK}(\mathbb{I}_n, A) & \xrightarrow{[\iota]\cdot} & \mathrm{KK}(\mathrm{S}\,M_n, A) \\ {\scriptstyle [\eta]\cdot}\uparrow & & & & \downarrow{\scriptstyle [\mathrm{S}\,\eta]\cdot} \\ \mathrm{KK}(M_n, A) & & & & \mathrm{KK}(\mathrm{S}\,\mathbb{C}, A) \\ \| & & & & \| \\ \mathrm{KK}(\mathrm{S}\,M_n, \mathrm{S}\,A) & \xleftarrow[{[\iota]\cdot}]{} & \mathrm{KK}(\mathbb{I}_n, \mathrm{S}\,A) & \xleftarrow[{[\delta_1]\cdot}]{} & \mathrm{KK}(\mathbb{C}, \mathrm{S}\,A) \end{array}$$

and KK-natural by associativity of the Kasparov product.

As the considered isomorphism $\mathrm{KK}(M_n, A) \to \mathrm{KK}(\mathbb{C}, A)$ is the one induced by the map $\mathrm{diag}(\mathrm{id}, 0, \ldots, 0)\colon \mathbb{C} \to M_n$, one sees that the map $[\eta]\colon \mathrm{KK}(M_n, A) \to \mathrm{KK}(\mathbb{C}, A)$ is but multiplication by n. ♡

Of course we desire to determine the natural transformation associated to the building blocks. As stated in [Eil95], we can determine them merely by composing $*$-homomorphisms.

Lemma 4.12 With respect to the chosen generators, the maps ρ_n^0 and β_n^0 associated to the building blocks are as follows: For $\mathbb{C}$ and $C(S^1)$ the map ρ_n^0 is the identity and the map β_n^0 is the zero map, while for $\mathbb{I}_m^\sim$ the map ρ_n^0 is represented by the matrix $\begin{pmatrix}1\\1\end{pmatrix}$ and the map β_n^0 by the matrix $\frac{m}{n}\begin{pmatrix}-1 & 1\end{pmatrix}$.

Proof. As $\mathrm{K}_0(\mathbb{C}; \mathbb{Z} \oplus \mathbb{Z}/n)$ is generated by $(1,0) = [\delta_0]$ and $(1,1) = [\delta_1]$, we get via the splitting mappings of 4.3 the generator $[\delta_0\eta] = [\mathrm{id}]$ for $\mathrm{K}_0(\mathbb{C})$ and the generator $[\delta_1\iota] - [\delta_0\iota] = [\delta_1\iota]$ for $\mathrm{K}_0(\mathbb{C}; \mathbb{Z}/n)$; for the groups $\mathrm{K}_0(C(S^1))$ and $\mathrm{K}_0(C(S^1); \mathbb{Z}/n)$ we get the generators $[\eta\delta_0\eta] = [\eta]$ respectively $[\eta\delta_1\iota]$; and for the groups $\mathrm{K}_0(\mathbb{I}_m^\sim)$ and $\mathrm{K}_0(\mathbb{I}_m^\sim; \mathbb{Z}/n)$ we get $[\eta\delta_0\eta] = [\eta]$ respectively $(1,1) = [\eta\delta_1\iota]$ and $(0, \frac{n}{(m,n)}) = [\mathrm{in}_{m,n}\iota]$.

Clearly, $[\mathrm{id}] \mapsto [\delta_1\iota]$ and $[\delta_1\iota] \mapsto [\delta_1\iota\iota] = 0$, hence $\rho_n^0 = 1$ and $\beta_n^0 = 0$ for $\mathbb{C}$. Similarly, as $[\eta] \mapsto [\eta\delta_1\iota]$ and $[\eta\delta_1\iota] \mapsto [\eta\delta_1\iota\iota] = 0$, $\rho_n^0 = 1$ and $\beta_n^0 = 0$ for $C(S^1)$.

Now the case $\mathbb{I}_m^\sim$: As $[\eta] \mapsto [\eta\delta_1\iota] = (1,1)$, we see that $\rho_n^0 = \begin{pmatrix}1\\1\end{pmatrix}$ for $\mathbb{I}_m^\sim$, and as $(1,1) = [\eta\delta_1\iota] \mapsto [\eta\delta_1\iota\iota] = 0$ and as $(0, \frac{n}{(n,m)}) = [\mathrm{in}_{m,n}\iota] \mapsto [\mathrm{in}_{m,n}\iota\iota] = \frac{n}{m}(n,m)[\varphi]$ where $\varphi\colon \mathrm{S}\,\mathbb{C} \to \mathbb{I}_m^\sim$ is $f \mapsto \mathrm{diag}(f, 0., \ldots, 0)$ and $[\varphi] \in \mathrm{KK}(\mathrm{S}\,\mathbb{C}, \mathbb{I}_m^\sim)$ corresponds to $[u_m]_1 \in \mathrm{K}_1(\mathbb{I}_m^\sim)$ via the isomorphism $\mathrm{KK}(\mathrm{S}\,\mathbb{C}, -) = \mathrm{K}_1(-)$, we get that $\beta_n^0 = \frac{m}{n}\begin{pmatrix}-1 & 1\end{pmatrix}$. ♡

4.3 The invariant $\mathbb{K}(-; n)$

We now define the invariant as a functor from the category of AD algebras to the category of complexes of abelian groups. Please cf. 4.11 and 4.3.

Definition 4.13 On the class of AD algebras, the invariant $\mathbb{K}(-; n)$ consists of the sequence

$$\mathrm{K}_0(-) \xrightarrow{\rho_n^0} \mathrm{K}_0(-; \mathbb{Z}/n) \xrightarrow{\beta_n^0} \mathrm{K}_1(-)$$

along with the scale $\Sigma(-) \subseteq \mathrm{K}_0(-)$ and the positive cones $\mathrm{K}_*(-)^+$ and $\mathrm{K}_0(-; \mathbb{Z} \oplus \mathbb{Z}/n)^+$.

As we will no longer be dealing with $\mathrm{K}_1(-; \mathbb{Z}/n)$, we will skip the superscript 0 in the naming of the maps ρ_n^0 and β_n^0.

Remark 4.14 In [Eil95] a more generalized invariant $\mathbb{K}(-; \infty)$ is defined as the limit of the inductive I-system of complexes $\mathbb{K}(-; n)$, I denoting the integers $\mathbb{N}$ ordered as $n \leq m$ when $n \mid m$. The connecting maps of this system is induced by the maps $[\mathrm{in}_{n,m}]\cdot\colon \mathrm{KK}(\mathbb{I}_n^\sim, A \otimes C(S^1)) \to \mathrm{KK}(\mathbb{I}_m^\sim, A \otimes C(S^1))$ via the splitting maps of 4.3 as

$\kappa^0_{m,n} = [\eta][\mathrm{in}_{n,m}][\delta_0]\cdot$, $\kappa_{m,n} = [\iota][\mathrm{in}_{n,m}][\mathrm{id}_{\mathbb{I}^{\sim}_n} - \eta\delta_0]\cdot$ and $\kappa^1_{m,n} = [\eta][\mathrm{in}_{n,m}][\delta_0]\cdot$. One must do some calculations to check that the diagram

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{\rho_n} & \mathrm{K}_0(A;\mathbb{Z}/n) & \xrightarrow{\beta_n} & \mathrm{K}_1(A) \\ \downarrow{\scriptstyle \kappa^0_{m,n}} & & \downarrow{\scriptstyle \kappa_{m,n}} & & \downarrow{\scriptstyle \kappa^1_{m,n}} \\ \mathrm{K}_0(A) & \xrightarrow{\rho_m} & \mathrm{K}_0(A;\mathbb{Z}/m) & \xrightarrow{\beta_m} & \mathrm{K}_1(A) \end{array}$$

commutes.

Since we have already determined $\mathbb{K}(A;n)$ when A is a building block, the following lemma gives us the power to – in theory – determine $\mathbb{K}(A;n)$ for any AD algebra A.

Lemma 4.15 The invariant $\mathbb{K}(-;n)$ is continuous. I.e. if $A = \varinjlim(A_i, f_i)$, then

$$\mathrm{K}_*(A;\mathbb{Z}\oplus\mathbb{Z}/n) = \varinjlim(\mathrm{K}_*(A_i;\mathbb{Z}\oplus\mathbb{Z}/n), \mathrm{K}_*(f_i;\mathbb{Z}\oplus\mathbb{Z}/n))$$

as $(\mathbb{Z}/2)^2$-graded groups with the positive cones preserved, as

$$\mathrm{K}_*(A)^+ = \bigcup_i \mathrm{K}_*(f_i)(\mathrm{K}_*(A_i)^+),$$

$$\mathrm{K}_0(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+ = \bigcup_i \mathrm{K}_0(f_i;\mathbb{Z}\oplus\mathbb{Z}/n)(\mathrm{K}_0(A_i;\mathbb{Z}\oplus\mathbb{Z}/n)^+),$$

as well as the scale, as $\Sigma(A) = \bigcup_i \mathrm{K}_0(f_i)(\Sigma(A_i))$, and with the diagram

$$\begin{array}{ccccc} \mathrm{K}_0(A_i) & \xrightarrow{\rho^{A_i}_n} & \mathrm{K}_0(A_i;\mathbb{Z}/n) & \xrightarrow{\beta^{A_i}_n} & K_1(A_i) \\ \downarrow{\scriptstyle \mathrm{K}_0(f_i)} & & \downarrow{\scriptstyle \mathrm{K}_0(f_i;\mathbb{Z}/n)} & & \downarrow{\scriptstyle \mathrm{K}_1(f_i)} \\ \mathrm{K}_0(A) & \xrightarrow{\rho^{A}_n} & \mathrm{K}_0(A;\mathbb{Z}/n) & \xrightarrow{\beta^{A}_n} & K_1(A) \end{array}$$

commuting for any i.

Proof. As the C^*-algebras $\mathbb{C}$, $\mathbb{I}_n$ and $\mathbb{I}^{\sim}_n$ are separable, nuclear, lie in the bootstrap class and have finitely generated K_*-groups, we may use 2.12 on any of the nine groups in the diagram that in 4.3 defines the $(\mathbb{Z}/2)^2$-grading. And as the diagram is KK-natural, the $(\mathbb{Z}/2)^2$-grading is respected. As the the maps in the diagram of 4.3 are positive, the positive cones are respected. And the proof of [RLL00, 6.3.2] shows that the scale is respected as stated. Commutativity of the diagram above follows from associativity of the Kasparov product. ♡

For later use, we determine $\mathbb{K}(\alpha;k)$ for a few $*$-homomorphisms α between building blocks.

Lemma 4.16 Consider the maps $\delta_0, \delta_1 \colon \mathbb{I}_n^\sim \to \mathbb{C}$ and $\mathrm{in}_{m,n}, \overline{\mathrm{in}}_{m,n} \colon \mathbb{I}_n^\sim \to M_{\frac{n}{(n,m)}}(\mathbb{I}_m^\sim)$. Consider also for $t \in (0,1)$ the map $\delta_t \colon \mathbb{I}_n^\sim \to M_n$ given by $f \mapsto f(t)$. For any k we get

$$\mathbb{K}(\delta_0; k) = (1, \begin{pmatrix}1 & 0\end{pmatrix}, 0) \text{ and } \mathbb{K}(\delta_1; k) = (1, \begin{pmatrix}0 & 1\end{pmatrix}, 0),$$

and we get

$$\mathbb{K}(\mathrm{in}_{m,n}; k) = (\frac{n}{(m,n)}, \frac{n}{(m,n)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \frac{m}{(n,m)})$$

and

$$\mathbb{K}(\overline{\mathrm{in}}_{m,n}; k) = (\frac{n}{(n,m)}, \frac{n}{(n,m)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, -\frac{m}{(n,m)}).$$

Furthermore,

$$\mathbb{K}(\delta_t; n) = (n, \begin{pmatrix}0 & 0\end{pmatrix}, 0).$$

Again, the matrix representations of the maps are with respect to our chosen generators for the groups.

Proof. For clarity we expand the notation by adding superscripts to specify the domains or codomains of the maps: $\iota^j \colon \mathbb{I}_j \to \mathbb{I}_j^\sim$, $\delta_i^j \colon \mathbb{I}_j^\sim \to \mathbb{C}$, $\delta_t^j \colon \mathbb{I}_j^\sim \to M_j$ and $\eta^j \colon \mathbb{C} \to \mathbb{I}_j^\sim$.

As $\mathrm{K}_0(\mathbb{C}; \mathbb{Z} \oplus \mathbb{Z}/k)$ is generated by $(1,0) = [\delta_0^k]$ and $(1,1) = [\delta_1^k]$, we get via the splitting maps of 4.3 the generator $[\delta_0^k \eta^k] = [\mathrm{id}]$ for $\mathrm{K}_0(\mathbb{C})$ and the generator $[\delta_1^k \iota^k] - [\delta_0^k \iota^k] = [\delta_1^k \iota^k]$ for $\mathrm{K}_0(\mathbb{C}; \mathbb{Z}/k)$; and for the groups $\mathrm{K}_0(\mathbb{I}_n^\sim)$ and $\mathrm{K}_0(\mathbb{I}_n^\sim; \mathbb{Z}/k)$ we get the generators $[\eta^n \delta_0^k \eta^k] = [\eta^n]$ respectively $(1,1) = [\eta^n \delta_1^k \iota^k]$ and $(0, \frac{k}{(n,k)}) = [\mathrm{in}_{n,k} \iota^k]$.

Clearly $\delta_i^n \eta^n = \mathrm{id}$, hence $\mathrm{K}_0(\delta_i^n) = 1$; and as $\mathrm{K}_1(\mathbb{C}) = 0$, $\mathrm{K}_1(\delta_i^n) = 0$. Also, as $\delta_i^n \eta^n \delta_1^k \iota^k = \delta_1^k \iota^k$ we obtain $\mathrm{K}_0(\delta_i^n; \mathbb{Z}/k)([\eta^n \delta_1^k \iota^k]) = [\delta_1^k \iota^k]$, and as $(\delta_i^n \otimes M_{\frac{k}{(n,k)}})\mathrm{in}_{n,k} = \mathrm{diag}(\delta_i^k, \ldots, \delta_i^k)$ we obtain $\mathrm{K}_0(\delta_i^n; \mathbb{Z}/k)([\mathrm{in}_{n,k} \iota^k]) = \frac{k}{(n,k)}[\delta_i^k \iota^k]$; and hereby we can conclude $\mathrm{K}_0(\delta_0^n; \mathbb{Z}/k) = \begin{pmatrix}1 & 0\end{pmatrix}$ and $\mathrm{K}_0(\delta_1^n; \mathbb{Z}/k) = \begin{pmatrix}0 & 1\end{pmatrix}$ as $\delta_0^k \iota^k = 0$.

Notice that $[\mathrm{in}_{m,n} \eta^n] = [\overline{\mathrm{in}}_{m,n} \eta^n] = [\eta^m \otimes M_{\frac{n}{(n,m)}}] = \frac{n}{(n,m)}[\eta^m]$, hence $\mathrm{K}_0(\mathrm{in}_{m,n}) = \mathrm{K}_0(\overline{\mathrm{in}}_{m,n}) = \frac{n}{(n,m)}$. As for $\mathrm{K}_0(\mathrm{in}_{m,n}; \mathbb{Z}/k)$ and $\mathrm{K}_0(\overline{\mathrm{in}}_{m,n}; \mathbb{Z}/k)$, we first notice that

$$[\mathrm{in}_{m,n} \eta^n \delta_1^k] = [\overline{\mathrm{in}}_{m,n} \eta^n \delta_1^k] = [\eta^m \delta_1^k \otimes M_{\frac{n}{(n,m)}}] = \frac{n}{(n,m)}[\eta^m \delta_1^k]$$

hence $\mathrm{K}_0(\mathrm{in}_{m,n}; \mathbb{Z}/k)(1,1) = \mathrm{K}_0(\overline{\mathrm{in}}_{m,n}; \mathbb{Z}/k)(1,1) = \frac{n}{(n,m)}(1,1)$, and second we see that

$$[(\mathrm{in}_{m,n} \otimes M_{\frac{k}{(n,k)}})\mathrm{in}_{n,k}] = [\mathrm{in}_{m,k} \otimes M_{\frac{n(m,k)}{(n,k)(n,m)}}]$$

$$[(\overline{\mathrm{in}}_{m,n} \otimes M_{\frac{k}{(n,k)}})\mathrm{in}_{n,k}] = [\overline{\mathrm{in}}_{m,k} \otimes M_{\frac{n(m,k)}{(n,k)(n,m)}}]$$

whereby it follows that $\mathrm{K}_0(\mathrm{in}_{m,n}; \mathbb{Z}/k)(0, \frac{k}{(n,k)}) = \frac{n(m,k)}{(n,k)(n,m)}(0, \frac{k}{(m,k)}) = \frac{n}{(n,m)}(0, \frac{k}{(n,k)})$ and similarly $\mathrm{K}_0(\overline{\mathrm{in}}_{m,n}; \mathbb{Z}/k)(0, \frac{k}{(n,k)}) = \frac{n}{(n,m)}(\frac{k}{(n,k)}, 0)$, and it hereby follows that the maps have the claimed matrix representations. As for $\mathrm{K}_1(\mathrm{in}_{m,n})$ and $\mathrm{K}_1(\overline{\mathrm{in}}_{m,n})$, we consider the generator $[u_n]_1$ and notice that $[\mathrm{in}_{m,n}(u_n)]_1 = \frac{m}{(n,m)}[\mathrm{diag}(u_m, 1_m, \ldots, 1_m)]_1 = \frac{m}{(n,m)}[u_m]_1$ and $[\overline{\mathrm{in}}_{m,n}(u_n)]_1 = \frac{m}{(n,m)}[u_m^*]_1 = -\frac{m}{(n,m)}[u_m^*]_1$, cf. 4.7.

Finally, as $\delta_t^n \eta^n = \operatorname{diag}(\mathrm{id}, \ldots, \mathrm{id})$, $\mathrm{K}_0(\delta_t^n) = n$; and as $\mathrm{K}_1(\mathbb{C}) = 0$, $\mathrm{K}_1(\delta_t^n) = 0$. Now, $\delta_t^n \eta^n \delta_1^n \iota^n = \operatorname{diag}(\delta_1^n \iota^n, \ldots, \delta_1^n \iota^n)$ which is null homotopic according to 3.1, hence $\mathrm{K}_0(\delta_t^n; \mathbb{Z}/n)([\eta^n \delta_1^n \iota^n]) = 0$. And $\delta_t^n \mathrm{in}_{n,n} \iota^n = \delta_t^n \iota^n$ is homotopic to 0 via the point-wise continuous path $s \mapsto \delta_{st}^n \iota^n$ of $*$-homomorphisms, so $\mathrm{K}_0(\delta_t^n; \mathbb{Z}/n)([\mathrm{in}_{n,n} \iota^n]) = 0$. Ergo, $\mathrm{K}_0(\delta_t^n; \mathbb{Z}/n) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. ♡

Remark 4.17 Via the unital embeddings $\mathbb{C} \to C(S^1)$ and $\mathbb{C} \to \mathbb{I}_m^\sim$, the homomorphisms δ_0 and δ_1 induce maps in $\operatorname{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(C(S^1); k))$ and $\operatorname{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(\mathbb{I}_m^\sim; k))$ of the form

$$(1, \begin{pmatrix} 1 & 0 \end{pmatrix}, 0) \quad \text{and} \quad (1, \begin{pmatrix} 0 & 1 \end{pmatrix}, 0),$$

and

$$(1, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 0) \quad \text{and} \quad (1, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, 0).$$

As we have decided to let η denote any unital embedding of $\mathbb{C}$, we will be naming both these pairs of maps $[\eta\delta_0]$ and $[\eta\delta_1]$, and hopefully this will not cause any confusion as we are never dealing with both $\operatorname{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(C(S^1); k))$ and $\operatorname{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(\mathbb{I}_m^\sim; k))$ at the same time.

5 Insufficiency of ordinary K-theory

In this section we construct two nonisomorphic real rank zero AD algebras B^0 and B^1 with $(\mathrm{K}_*(B^0), \mathrm{K}_*(B^0)^+, \Sigma(B^0))$ and $(\mathrm{K}_*(B^1), \mathrm{K}_*(B^1)^+, \Sigma(B^1))$ being identical, thus showing that ordinary K-theory is insufficient when it comes to classifying real rank zero AD algebras.

The example also shows the strength of K-theorical invariants, as we show that B^0 and B^1 are nonisomorphic by showing that the ordered groups $\mathrm{K}_0(B^0; \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_0(B^1; \mathbb{Z} \oplus \mathbb{Z}/n)$ are nonisomorphic, ordered groups being much easier to deal with than C^*-algebras.

Lemma 5.1 Given $n, m \in \mathbb{N}$ with $m \neq 1$, consider the subgroup

$$G = \left\{ (x, (y_i)) \in \mathbb{Z}[\tfrac{1}{m}] \oplus \prod_{i \in \mathbb{N}} \mathbb{Z} \;\middle|\; y_i = nm^i x \text{ eventually} \right\}$$

of $\mathbb{R}^{\mathbb{N}}$, equipped with the standard point-wise order on $\mathbb{R}^{\mathbb{N}}$. Then the order-automorphisms of G are exactly those of the form

$$\varphi_\sigma \colon (x, (y_i)) \mapsto (x, (y_{\sigma(i)}))$$

for some permutation $\sigma \in \{\tau \in \mathrm{S}_{\mathbb{N}} \mid \tau(i) = i \text{ eventually}\}$.

Proof. Clearly, any such φ_σ is an order-automorphism. Now, let $\varphi \colon G \to G$ be an order-automorphism, and let us construct $\sigma \in \mathrm{S}_{\mathbb{N}}$ such that $\sigma(i) = i$ eventually and $\varphi = \varphi_\sigma$. Define

$$d_j = (0, (\delta_{ij})) \in G,$$

where δ here denotes the Kronecker delta. Notice that the minimal positive elements of G – i.e. those $g \in G^+$ for which $0 \leq \tilde{g} < g$ implies $0 = \tilde{g}$ – are $\{d_j \mid j \in \mathbb{N}\}$. As φ is an order-automorphism, $\varphi(\{d_j \mid j \in \mathbb{N}\}) = \{d_j \mid j \in \mathbb{N}\}$, and we can therefore define a permutation $\sigma \in \mathrm{S}_{\mathbb{N}}$ by $\varphi(d_j) = d_{\sigma^{-1}(j)}$. Notice that by this definition, $\varphi(d_j) = (0, (\delta_{i\sigma^{-1}(j)})) = (0, (\delta_{\sigma(i)j})) = \varphi_\sigma(d_j)$.

Consider

$$e = (1, (nm^i)) = (1, nm, nm^2, nm^3, nm^4, \ldots) \in G.$$

We desire to show that $\varphi(e) = (1, (nm^{\sigma(i)}))$. Denote $\varphi(e) = (w, (z_i))$, and notice as $(w, (z_i)) \geq 0$ that for each $j \in \mathbb{N}$, $e \geq nm^{\sigma(j)} d_{\sigma(j)}$ and $(w, (z_i)) \geq z_j d_j$. From the first inequality we obtain

$$(w, (z_i)) = \varphi(e) \geq \varphi(nm^{\sigma(j)} d_{\sigma(j)}) = nm^{\sigma(j)} d_j,$$

hence $z_i \geq nm^{\sigma(j)} \delta_{ij}$ for each i, in particular $z_j \geq nm^{\sigma(j)}$. And from the second we obtain

$$e = \varphi^{-1}(w, (z_i)) \geq \varphi^{-1}(z_j d_j) = z_j d_{\sigma(j)},$$

hence $nm^i \geq z_j \delta_{i\sigma(j)}$ for all i, in particular $nm^{\sigma(j)} \geq z_j$. Ergo $z_j = nm^{\sigma(j)}$ for all j.

As $\varphi(e) \in G$, we can find an i_0 such that $nm^{\sigma(i)} = nm^i w$ when $i \geq i_0$. In particular, $nm^{\sigma(i_0+k)} = nm^{i_0+k}w = nm^{\sigma(i_0)+k}$, hence $\sigma(i_0+k) = \sigma(i_0)+k$ for all $k \geq 0$, ergo

$$\sigma(\{i_0, i_0+1, i_0+2, i_0+3, \dots\}) = \{\sigma(i_0), \sigma(i_0)+1, \sigma(i_0)+2, \dots\}$$

so as σ is bijective,

$$\sigma(\{1, 2, \dots, i_0-1\}) = \{1, 2, \dots, \sigma(i_0)-1\},$$

hence $\sigma(i_0) = i_0$. From this, it follows that $\sigma(i) = i$ when $i \geq i_0$, and that $w = 1$ as $nm^{i_0} = nm^{i_0}w$ and $\mathbb{Z}[\frac{1}{m}]$ is torsion-free.

Now, we have seen that $\varphi(d_j) = \varphi_\sigma(d_j)$ and $\varphi(e) = \varphi_\sigma(e)$. Let

$$z = (x, y_1, \dots, y_k, xnm^{k+1}, xnm^{k+2}, \dots) \in G$$

be given, and let us conclude that $\varphi(z) = \varphi_\sigma(z)$. Now, as

$$nm^{k+1}z = xnm^{k+1}e + \sum_{j=1}^{k}(y_j - xnm^j)nm^{k+1}d_j,$$

we see that $nm^{k+1}\varphi(z) = nm^{k+1}\varphi_\sigma(z)$, hence $\varphi(z) = \varphi_\sigma(z)$ as G is torsion-free. ♡

Example 5.2 Let $n \in \mathbb{N}\backslash\{1\}$, define $m = n+1$, and let $t \in (0,1)$ be given. Define maps $g^0, g^1 \colon \mathbb{I}_n^\sim \to M_n$ by $g^0 = \operatorname{diag}(\delta_0, \dots, \delta_0)$ and $g^1 = \operatorname{diag}(\delta_0, \delta_1, \dots, \delta_1)$, and $f \colon \mathbb{I}_n^\sim \to M_m(\mathbb{I}_n^\sim)$ by $f = \operatorname{diag}(\mathrm{id}, (M_n \otimes \eta)\delta_t)$.

By 4.16, $\mathbb{K}((M_n \otimes \eta)\delta_t; n) = (n, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0)$, and therefore

$$\mathbb{K}(g^l; n) = (n, \gamma^l, 0) \quad \text{and} \quad \mathbb{K}(f; n) = (m, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1)$$

with $\gamma^0 = \begin{pmatrix} 0 & 0 \end{pmatrix}$ and $\gamma^1 = \begin{pmatrix} 1 & n-1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}$.

Define for $l \in \{0,1\}$ C^*-algebras A_i^l as

$$A_1^l = \mathbb{I}_n^\sim, \quad A_i^l = M_{m^{i-1}}(\mathbb{I}_n^\sim) \oplus M_n \oplus M_{nm} \oplus \cdots \oplus M_{nm^{i-2}}$$

and maps $f_i^l \colon A_i^l \to A_{i+1}^l$ as

$$f_1^l(a) = (f(a), g^l(a)), \quad f_i^l(a_1, \dots, a_i) = ((M_{m^{i-1}} \otimes f)(a_1), a_2, \dots, a_i, (M_{m^{i-1}} \otimes g^l(a_1)),$$

and define $A^0 = \varinjlim(A_i^0, f_i^0)$ and $A^1 = \varinjlim(A_i^1, f_i^1)$. We now desire to determine the ordered groups $\mathrm{K}_0(A^l; \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_*(A^l)$.

The inductive systems $(G_i^l, g_i^l) = (\mathrm{K}_0(A_i^l), \mathrm{K}_0(f_i^l))$ are as follows:

$$G_i^l = \mathbb{Z}^i, \quad g_i^l(x_1, \dots, x_i) = (mx_1, x_2, \dots, x_i, nx_1)$$

By defining

$$G = \left\{ (x, (y_i)) \in \mathbb{Z}[\tfrac{1}{m}] \oplus \prod_{i \in \mathbb{N}} \mathbb{Z} \;\middle|\; y_i = nm^i x \text{ eventually} \right\}$$

and maps $\varphi_i^l \colon G_i^l \to G$ as

$$(x_1, \ldots, x_i) \mapsto (\frac{x_1}{m^i}, x_2, \ldots, x_i, nx_1, nmx_1, nm^2x_1, \ldots),$$

we see as $\varphi_i^l = \varphi_{i+1}^l g_i^l$, and as $\ker \varphi_i^l = 0 \subseteq \ker g_{\infty,i}^l$ and $\bigcup_i \operatorname{im} \varphi_i^l = G$, that $(G, \varphi_i^l) = \varinjlim (G_i^l, g_i^l)$.

As for $\mathrm{K}_1(A^l)$, the inductive systems $(\mathrm{K}_1(A_i^l), \mathrm{K}_1(f_i^l))$ are the system

$$\mathbb{Z}/n \xrightarrow{1} \mathbb{Z}/n \xrightarrow{1} \mathbb{Z}/n \xrightarrow{1} \mathbb{Z}/n \xrightarrow{1} \mathbb{Z}/n \xrightarrow{1} \cdots,$$

hence $\mathrm{K}_1(A^l) = \mathbb{Z}/n$. Notice that the systems $(\mathrm{K}_*(A_i^l), \mathrm{K}_*(f_i^l))$ have large denominators as, using the notation of 3.6, $\mathrm{K}_0(\pi_r)\,\mathrm{K}_0(f_i^l)\,\mathrm{K}_0(\iota_s) = m \geq 3$ when $\mathrm{K}_1(\pi_r)\,\mathrm{K}_1(f_i^l)\,\mathrm{K}_1(\iota_s) \neq 0$.

The inductive systems $(H_i^l, h_i^l) = (\mathrm{K}_0(A_i^l; \mathbb{Z}/n), \mathrm{K}_0(f_i^l; \mathbb{Z}/n))$ are

$$H_i^l = (\mathbb{Z}/n)^{i+1}, \quad h_i^l(x_1, \ldots, x_{i+1}) = (x_1, \ldots, x_{i+1}, l(x_1 - x_2)).$$

Define groups

$$H^l = \left\{ (a, b, (c_i)) \in \mathbb{Z}/n \oplus \mathbb{Z}/n \oplus \prod_{i \in \mathbb{N}} \mathbb{Z}/n \;\middle|\; c_i = l(a - b) \text{ eventually} \right\}$$

and maps $\psi_i^l \colon H_i^l \to H^l$ as

$$(x_1, \ldots, x_{i+1}) \mapsto (x_1, \ldots, x_{i+1}, l(x_1 - x_2), l(x_1 - x_2), \ldots).$$

As $\ker \psi_i^l = 0 \subseteq \ker h_{\infty,i}^l$ and $\bigcup_i \operatorname{im} \psi_i^l = H^l$, $(H^l, \psi_i^l) = \varinjlim (H_i^l, h_i^l)$.

Let us now determine the positive cones in $\mathrm{K}_0(A^l; \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_*(A^l)$. As $\mathrm{K}_*(A_i^l)^+ = \{(x_1, \ldots, x_i, x) \in \mathbb{Z}^i \oplus \mathbb{Z}/n \mid x_k \geq 0, (x_1 \neq 0 \vee x = 0)\}$, we see that

$$(G \oplus \mathbb{Z}/n)^+ = \bigcup_{i \in \mathbb{N}} (\varphi_i^l \oplus \mathrm{id}_{\mathbb{Z}/n})(\mathrm{K}_*(A_i^l)^+) = \{(x, (y_i), z) \mid x \geq 0, y_i \geq 0, (x \neq 0 \vee z = 0)\}.$$

And as $\mathrm{K}_0(A_i^l; \mathbb{Z} \oplus \mathbb{Z}/n)^+$ equals

$$\{(x_1, \ldots, x_i, z_1, \ldots, z_{i+1}) \in \mathbb{Z}^i \oplus (\mathbb{Z}/n)^{i+1} \mid x_1 \geq z_1 \geq 0, x_1 \geq z_2 \geq 0, x_k \geq z_{k+1} \geq 0\},$$

we get that

$$\begin{aligned}(G \oplus H^l)^+ &= \bigcup_{i \in \mathbb{N}} (\varphi_i^l \oplus \psi_i^l)(\mathrm{K}_0(A_i^l; \mathbb{Z} \oplus \mathbb{Z}/n)^+) \\ &= \{(x, (y_i), a, b, (c_i)) \mid x \geq 0, (x \neq 0 \vee a = b = 0), y_i \geq c_i \geq 0\}.\end{aligned}$$

Lemma 5.3 Consider the two C^*-algebras A^0 and A^1 defined in 5.2. Then the ordered groups $\mathrm{K}_0(A^0;\mathbb{Z}\oplus\mathbb{Z}/n)$ and $\mathrm{K}_0(A^1;\mathbb{Z}\oplus\mathbb{Z}/n)$ are not isomorphic.

Proof. Assume, seeking a contradiction, that $(\varphi_0,\varphi)\colon \mathrm{K}_0(A^0;\mathbb{Z}\oplus\mathbb{Z}/n)\to\mathrm{K}_0(A^1;\mathbb{Z}\oplus\mathbb{Z}/n)$ is an order-isomorphism.

As in 5.2, we let G denote the ordered group $\mathrm{K}_0(A^0)=\mathrm{K}_0(A^1)$, and let H^0 and H^1 denote the ordered groups $\mathrm{K}_0(A^0;\mathbb{Z}/n)$ and $\mathrm{K}_0(A^1;\mathbb{Z}/n)$ respectively. See 5.2 for descriptions of G, H^0 and H^1, and of the positive cone in $G\oplus H^0$ and in $G\oplus H^1$.

First of all, we notice that we may assume that $\varphi_0=\mathrm{id}$. According to 5.1, there exists a permutation $\sigma\in\mathrm{S}_\mathbb{N}$ so that $\varphi_0(x,(y_i))=(x,(y_{\sigma(i)})$ for all $(x,(y_i))\in G$. By defining $\tilde{\varphi}\colon H^0\to H^0$ by $(a,b,(c_i))\mapsto(a,b,(c_{\sigma(i)}))$, we see that $(\varphi_0^{-1},\tilde{\varphi})\colon G\oplus H^0\to G\oplus H^0$ is an order-automorphism. Hence $(\varphi_0\varphi_0^{-1},\varphi\tilde{\varphi})$ is an order-isomorphism.

Ergo, without loss of generality, we may assume that we have an order-isomorphism of the form $(\mathrm{id},\varphi)\colon G\oplus H^0\to G\oplus H^1$. To reach the desired contradiction, let us show that φ cannot be injective. Define for each $j\in\mathbb{N}$ an element $x_j\in G$, and define $e_1,e_2\in H^0$, as

$$\begin{aligned} x_j &= (1,(nm^i(1-\delta_{ij}))),\\ e_1 &= (1,0,0,0,0,0,0,0,0,\ldots),\\ e_2 &= (0,1,0,0,0,0,0,0,0,\ldots),\end{aligned}$$

where δ is the Kronecker delta. Write $\varphi(e_k)=(a_k,b_k,(c_i^k))$, using minimal representatives. For any $j\in\mathbb{N}$, we see as (x_j,e_1) and (x_j,e_2) are positive, that $(x_j,\varphi(e_1))$ and $(x_j,\varphi(e_2))$ are positive. Hence, for all i,j, $nm^i(1-\delta_{ij})\geq c_i^k\geq 0$, so by choosing $i=j$, we see that $c_i^k=0$ for all i. Now, as $\varphi(e_k)\in H^1$, $c_i^k=a_k-b_k$ eventually, hence $a_k=b_k$.

Consider the elements a_2e_1 and a_1e_2 in H^0. Clearly, $a_2e_1\neq a_1e_2$, but

$$\varphi(a_2e_1)=a_2(a_1,a_1,0,0,0,\ldots)=a_1(a_2,a_2,0,0,0,\ldots)=\varphi(a_1e_2).$$

♡

The proposition below improves the above result by adding the requirement that the C^*-algebras are of real rank zero.

Proposition 5.4 Consider the two C^*-algebras A^0 and A^1 defined in 5.2. Then there exist real rank zero AD algebras B^0 and B^1 where for each $l\in\{0,1\}$ the ordered group $\mathrm{K}_0(B^l;\mathbb{Z}\oplus\mathbb{Z}/n)$ is isomorphic to $\mathrm{K}_0(A^l;\mathbb{Z}\oplus\mathbb{Z}/n)$ and the ordered group $\mathrm{K}_*(B^l)$ isomorphic to $\mathrm{K}_*(A^l)$. In particular, $\mathrm{K}_*(B^0)$ and $\mathrm{K}_*(B^1)$ are isomorphic as ordered groups, and $\Sigma(B^0)=\Sigma(B^1)$, but B^0 and B^1 are not isomorphic.

Proof. As the inductive systems $(\mathrm{K}_*(A_i^l),\mathrm{K}_*(f_i^l))$ have large denominators, we get by 3.9 inductive systems $(A_{i_k}^l,g_k^l)$ with real rank zero limits and satisfying for each $l\in\{0,1\}$ that $(A_{i_k}^l,g_k^l)$ is KK-shape equivalent to (A_i^l,f_i^l). Denote $B^l=\varinjlim(A_{i_k}^l,g_k^l)\otimes K(\ell^2(\mathbb{N}))$.

By continuity of $\mathbb{K}(-;n)$, cf. 4.15, we see by arguing as in 3.11 that the ordered group $\mathrm{K}_0(B^l;\mathbb{Z}\oplus\mathbb{Z}/n)$ is isomorphic to $\mathrm{K}_0(A^l;\mathbb{Z}\oplus\mathbb{Z}/n)$ and the ordered group $\mathrm{K}_*(B^l)$

isomorphic to $\mathrm{K}_*(A^l)$. By 5.2 we conclude that $\mathrm{K}_*(B^0)$ and $\mathrm{K}_*(B^1)$ are isomorphic, by 5.3 we see that $\mathrm{K}_0(B^0, \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_0(B^1; \mathbb{Z} \oplus \mathbb{Z}/n)$ are not, and as B^0 and B^1 are stabilized, $\Sigma(B^0) = \Sigma(B^1)$. ♡

6 Taking advantage of the UCT

In this section we prove the main theorem, namely that under some conditions on the AD algebras A and B, any isomorphism between $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$ may be lifted to an isomorphism between A and B. Firstly, we therefore define the notion of $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$ being isomorphic.

Definition 6.1 By $\mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$ we denote the set of triplets $(\varphi_0, \varphi, \varphi_1)$ of group homomorphisms fitting into the commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{\rho_n^A} & \mathrm{K}_0(A;\mathbb{Z}/n) & \xrightarrow{\beta_n^A} & K_1(A) \\ \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_1} \\ \mathrm{K}_0(B) & \xrightarrow{\rho_n^B} & \mathrm{K}_0(B;\mathbb{Z}/n) & \xrightarrow{\beta_n^B} & K_1(B). \end{array}$$

We let $\mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))^+$ denote the subset consisting of the triplets for which (φ_0, φ) and (φ_0, φ_1) are positive maps. And by $\mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))^{+,\Sigma}$ we denote the subset of positive triplets for which $\varphi_0(\Sigma(A)) \subseteq \Sigma(B)$. A triplet $(\varphi_0, \varphi, \varphi_1)$ is said to be N-large if (φ_0, φ_1) is. We consider $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$ isomorphic if there exists $(\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))^{+,\Sigma}$ with the maps φ_0, φ and φ_1 being bijective and $(\varphi_0^{-1}, \varphi^{-1}, \varphi_1^{-1}) \in \mathrm{Hom}(\mathbb{K}(B;n), \mathbb{K}(A;n))^{+,\Sigma}$.

As the maps ρ_n and β_n are defined as Kasparov multiplication from the left with certain KK-classes, there is for every pair A, B of AD algebras a positive homomorphism

$$\Gamma^n\colon\ \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$$

given by Kasparov multiplication from the right. By the Universal Coefficient Theorem (2.11) we notice that for any pair of maps $(\varphi_0, \varphi_1)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ there is a map $\varphi\colon \mathrm{K}_0(A;\mathbb{Z}/n) \to \mathrm{K}_0(B;\mathbb{Z}/n)$ such that $(\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$.

6.1 On building blocks

It turns out that we can say quite a lot about the map Γ^n in the case where A and B are finite direct sums of (matrix algebras over) building blocks. All proofs are from [Eil95]. We start out with A and B just being building blocks, first the case $A = C(S^1)$ and then the case $A = \mathbb{I}_n^\sim$. The last case is annoyingly technical on the level of minor details.

Lemma 6.2 If $\mathrm{K}_*(A)$ is free or $\mathrm{K}_*(B)$ is divisible, then the map

$$\Gamma^n\colon\ \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$$

is an isomorphism.

Proof. In either case, $\mathrm{Ext}^1(\mathrm{K}_*(A), \mathrm{K}_{*+1}(B)) = 0$ whereby injectivity of Γ^n (and in the case $n = 1$ also surjectivity) follows from the UCT (2.11).

As for surjectivity when $n \geq 2$, let $(\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$ be given. By UCT we get an $\alpha \in \mathrm{KK}(A,B)$ such that $\Gamma^n(\alpha) = (\varphi_0, \tilde{\varphi}, \varphi_1)$. By exactness of the rows in the following commutative diagram

$$\begin{array}{ccccccccc} \mathrm{K}_0(A) & \xrightarrow{n} & \mathrm{K}_0(A) & \xrightarrow{\rho_n^A} & \mathrm{K}_0(A;\mathbb{Z}/n) & \xrightarrow{\beta_n^A} & K_1(A) & \xrightarrow{n} & K_1(A) \\ & & \downarrow{\scriptstyle \varphi_0} & & {\scriptstyle \varphi}\downarrow\downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \varphi_1} & & \\ \mathrm{K}_0(B) & \xrightarrow{n} & \mathrm{K}_0(B) & \xrightarrow{\rho_n^B} & \mathrm{K}_0(B;\mathbb{Z}/n) & \xrightarrow{\beta_n^B} & K_1(B) & \xrightarrow{n} & K_1(B), \end{array}$$

we will obtain $\tilde{\varphi} = \varphi$. If $\mathrm{K}_*(A)$ is free, then $n\colon \mathrm{K}_1(A) \to \mathrm{K}_1(A)$ is injective and ρ_n^A therefore surjective. As $\varphi\rho_n^A = \rho_n^B\varphi_0 = \tilde{\varphi}\rho_n^A$, $\varphi = \tilde{\varphi}$ then follows. If $\mathrm{K}_*(B)$ is divisible, then $n\colon \mathrm{K}_0(B) \to \mathrm{K}_0(B)$ is surjective and β_n^N therefore injective. As $\beta_n^B\varphi = \varphi_1\beta_n^A = \beta_n^B\tilde{\varphi}$, $\varphi = \tilde{\varphi}$ then follows. ♡

Lemma 6.3 Given $k, n, m \in \mathbb{N}$ with $n, m \neq 1$, the map

$$\Gamma^k\colon \mathrm{KK}(\mathbb{I}_n^\sim, \mathbb{I}_m^\sim) \to \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$$

is surjective. The map Γ^k is injective if and only if $n \mid k$. Furthermore, given any $\alpha \in \mathrm{KK}(\mathbb{I}_n^\sim, \mathbb{I}_m^\sim)$ one can find $x, y, d \in \mathbb{Z}$ with $\frac{m}{(n,m)} \mid y$ and $0 \leq d < n$ such that

$$\Gamma^k(\alpha) = (x, \begin{pmatrix} x & 0 \\ x - \frac{ny}{m} & \frac{ny}{m} \end{pmatrix} + d\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$$

for all $k \geq 2$, and for a fixed k the factor d will be unique modulo (n,k).

Proof. Recall from 4.7, 4.9 and 4.12 that the diagram an element $\Phi = (\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$ must make commute, is the following

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\binom{1}{1}} & \mathcal{Z}(n;k) & \xrightarrow{\frac{n}{k}(-1 \quad 1)} & \mathbb{Z}/n \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} \\ \mathbb{Z} & \xrightarrow{\binom{1}{1}} & \mathcal{Z}(m;k) & \xrightarrow{\frac{m}{k}(-1 \quad 1)} & \mathbb{Z}/m \end{array}$$

where $\mathcal{Z}(l;k) = \{(x,y) \in \mathbb{Z}/k \oplus \mathbb{Z}/k \mid x \equiv y \pmod{\frac{k}{(l,k)}}\}$. By UCT (2.11) we can find $\alpha \in \mathrm{KK}(\mathbb{I}_n^\sim, \mathbb{I}_m^\sim)$ such that $\Phi - \Gamma^k(\alpha) = (0, \tilde{\varphi}, 0)$ for some $\tilde{\varphi}$.

For any $d \in \mathbb{Z}$ we consider the map

$$\psi_d = d\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} : \mathcal{Z}(n;k) \to \mathcal{Z}(m;k)$$

and notice that it is not only well-defined but also $(0, \psi_d, 0) \in \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$. Clearly $\psi_d = \psi_e$ exactly when $d \equiv e \pmod{(n,k)}$. By 4.16 we see that $\psi_d = \Gamma^k(d([\eta\delta_1] -$

$[\eta\delta_0])) \in \operatorname{im}\Gamma^k$. Now, we wish for $\tilde{\varphi} = \psi_d$ for some d. Notice that $(1,1)$ and $(0, \frac{k}{(n,k)})$ generate $\mathcal{Z}(n;k)$, so if $\tilde{\varphi}((0, \frac{k}{(n,k)})) = (b,c)$ then

$$\tilde{\varphi} = \begin{pmatrix} -\frac{(n,k)b}{k} & \frac{(n,k)b}{k} \\ -\frac{(n,k)c}{k} & \frac{(n,k)c}{k} \end{pmatrix}$$

as $\tilde{\varphi}((1,1)) = 0$. Now, as $\frac{m}{k}(c-b) = 0$ in $\mathbb{Z}/m$, we conclude that $k \mid (c-b)$. And as $(0, \frac{k}{(n,k)})$ is of order (n,k) in $\mathcal{Z}(n;k)$, we conclude that (n,k) annihilates (b,c), hence $\frac{k}{(n,k)} \mid b$. We may therefore write $b = d\frac{k}{(n,k)}$ and $c = e\frac{k}{(n,k)}$ where $d \equiv e \pmod{(n,k)}$, and it follows that $\tilde{\varphi} = \psi_d = \psi_e$.

Hence $\Phi - \Gamma^k(\alpha) \in \operatorname{im}\Gamma^k$, and we may conclude that Γ^k is surjective. Furthermore, since $\mathrm{K}_*(\eta\delta_0)$ and $\mathrm{K}_*(\mathrm{in}_{m,n})$ generate $\mathrm{Hom}(\mathrm{K}_*(\mathbb{I}_n^\sim), \mathrm{K}_*(\mathbb{I}_m^\sim))$, cf. 4.16, we may conclude that $\mathbb{K}(\eta\delta_0;k)$, $\mathbb{K}(\eta\delta_1;k)$ and $\mathbb{K}(\mathrm{in}_{m,n};k)$ generate $\mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$.

Now, if $n \mid k$, then $(n,k) = n$ and $\mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$ therefore contains at least $(n,m)n$ torion elements, namely those of the form $\Gamma^k(\alpha_l) + \psi_d$ where $\alpha_l \in \mathrm{KK}(\mathbb{I}_n^\sim, \mathbb{I}_m^\sim)$ is a lift of $(0, \frac{m}{(n,m)}l)\colon \mathrm{K}_*(\mathbb{I}_n^\sim) \to \mathrm{K}_*(\mathbb{I}_m^\sim)$ and $(l,d) \in \{1,\dots,(n,m)\} \times \{1,\dots,n\}$, and at least one non-torsion element, namely $\Gamma^k(\alpha)$ where α is a lift of $(1,0) \in \mathrm{Hom}(\mathrm{K}_*(\mathbb{I}_n^\sim), \mathrm{K}_*(\mathbb{I}_m^\sim))$. As Γ^k is surjective and maps from a group which by UCT (2.11) is isomorphic to the group $\mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}/(n,m) = \mathbb{Z}/n \oplus \mathbb{Z} \oplus \mathbb{Z}/(n,m)$ and therefore has exactly one free part and exactly $(n,m)n$ torsion elements, Γ^k must be injective.

On the other hand, if n doesn't divide k, then $(n,k) < n$ hence $\psi_{(n,k)}\colon \mathcal{Z}(n;n) \to \mathcal{Z}(m;n)$ is non-zero and therefore $(n,k)([\eta\delta_1] - [\eta\delta_0]) \neq 0$. However $\Gamma^k((n,k)([\eta\delta_1] - [\eta\delta_0])) = \psi_{(n,k)} = 0$ as $(n,k) \equiv 0 \pmod{(n,k)}$, hence Γ^k is not injective.

As for the last claim of the lemma, given an $\alpha \in \mathrm{KK}(\mathbb{I}_n^\sim, \mathbb{I}_m^\sim)$ we notice that $\Gamma^n(\alpha) = (\varphi_0, \varphi, \varphi_1)$ where the part (φ_0, φ_1) must have the form (x,y) with $\frac{m}{(n,m)} \mid y$ as $\varphi_1\colon \mathbb{Z}/n \to \mathbb{Z}/m$. By not too many difficulties one can show that

$$\tilde{\varphi} = \begin{pmatrix} x & 0 \\ x - \frac{ny}{m} & \frac{ny}{m} \end{pmatrix}$$

defines a map $\tilde{\varphi}\colon \mathcal{Z}(n;k) \to \mathcal{Z}(m;k)$, and one easily checks that $(\varphi_0, \tilde{\varphi}, \varphi_1)$ lies in $\mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(\mathbb{I}_m^\sim;k))$, hence $\Gamma^n(\alpha) - (\varphi_0, \tilde{\varphi}, \varphi_1) = (0, \psi_d, 0)$ where we may choose $0 \le d < (n,n) = n$. Now, define $\tilde{\alpha} = (x - \frac{yn}{m})[\eta\delta_0] + \frac{y(n,m)}{m}[\mathrm{in}_{m,n}] + d([\eta\delta_1] - [\eta\delta_0])$. As $\Gamma^k(\tilde{\alpha}) = (x, \begin{pmatrix} x & 0 \\ x - \frac{ny}{m} & \frac{ny}{m} \end{pmatrix} + d\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$ for any $k \ge 2$, we see in particular that $\Gamma^n(\alpha) = \Gamma^n(\tilde{\alpha})$ whereby we conclude that $\alpha = \tilde{\alpha}$ as Γ^n is injective, and the desired follows. ♡

Quite similarly, also using 4.16, one can show the following.

Lemma 6.4 Given $k, n \in \mathbb{N}$ with $n \neq 1$, the map

$$\Gamma^k\colon \mathrm{KK}(\mathbb{I}_n^\sim, C(S^1)) \to \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k), \mathbb{K}(C(S^1);k))$$

is surjective. The map Γ^k is injective if and only if $n \mid k$. Furthermore, given any $\alpha \in \mathrm{KK}(\mathbb{I}_n^\sim, C(S^1))$ one can find $x, d \in \mathbb{Z}$ with $0 \le d < n$ such that

$$\Gamma^k(\alpha) = (x, \begin{pmatrix} x & 0 \end{pmatrix} + d \begin{pmatrix} -1 & 1 \end{pmatrix}, 0)$$

for all $k \ge 2$, and for a fixed k the factor d will be unique modulo (n, k).

As we want to lift isomorphisms in $\mathrm{Hom}(\mathbb{K}(A;k), \mathbb{K}(B;k))$ to isomorphisms $A \to B$, it is of interest for us to lift positive homomorphisms in $\mathrm{Hom}(\mathbb{K}(A;k), \mathbb{K}(B;k))$ to elements in $\mathrm{KK}(A,B)^+$. We start out with a special case and then use it to prove the claim in the general case.

Lemma 6.5 Given $\Phi = (\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(A;k))^+$ with $A \in \{\mathbb{I}_m^\sim, C(S^1)\}$ and $\varphi_1 = 0$, one can find $a_0, a_1 \in \mathbb{N}_0$ such that $\Gamma^k(a_0[\eta\delta_0] + a_1[\eta\delta_1]) = \Phi$.

Proof. Consider first the case $A = \mathbb{I}_m^\sim$. According to 6.3 and as $y = 0$, Φ is of the form

$$\Phi = (x, \begin{pmatrix} x-d & d \\ x-d & d \end{pmatrix}, 0)$$

where we may choose $0 \le d < (n,k)$. We notice that the positive element $(\frac{k}{(k,n)}, 0, \frac{k}{(k,n)}) \in \mathrm{K}_0(\mathbb{I}_n^\sim; \mathbb{Z} \oplus \mathbb{Z}/k)^+$ is mapped by (φ_0, φ) to the element $(\frac{xk}{(k,n)}, \frac{dk}{(k,n)}, \frac{dk}{(k,n)})$ of $\mathrm{K}_0(\mathbb{I}_m^\sim; \mathbb{Z}/ \oplus \mathbb{Z}/k)$ which must be positive as (φ_0, φ) is positive, whereby we conclude that $x \ge d$ as $0 \le d < k$, cf. 4.9 and the definition of $\mathbb{Z} \oplus_{\ge} \mathcal{Z}(m;k)$. By setting $a_0 = x - d$ and $a_1 = d$ we therefore obtain $a_0, a_1 \in \mathbb{N}_0$, and $\Gamma^k(a_0[\eta\delta_0] + a_1[\eta\delta_1]) = \Phi$ according to 4.16.

The case $A = C(S^1)$ is dealt with almost identically. By 6.4 we can find $0 \le d < (n,k)$ such that

$$\Phi = (x, \begin{pmatrix} x & 0 \end{pmatrix} + d \begin{pmatrix} -1 & 1 \end{pmatrix}, 0),$$

and as $(\frac{k}{(k,n)}, 0, \frac{k}{(k,n)})$ is mapped to $(\frac{xk}{(n,k)}, \frac{dk}{(n,k)})$ we again obtain $d \le x$ as the group $\mathrm{K}_0(C(S^1); \mathbb{Z} \oplus \mathbb{Z}/k)$ is isomorphic to $\mathbb{Z} \oplus_{\ge} \mathbb{Z}/k$. By setting $a_0 = x - d$ and $a_1 = d$ we obtain the desired, cf. 4.16. ♡

Lemma 6.6 If n divides k, then given $\Phi \in \mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim; k), \mathbb{K}(\mathbb{I}_m^\sim; k))^+$ one can find $a_0, a_1, a_2, a_3 \in \mathbb{N}_0$ such that $\Gamma^k(a_0[\eta\delta_0] + a_1[\eta\delta_1] + a_2[\mathrm{in}_{m,n}] + a_3[\overline{\mathrm{in}}_{m,n}]) = \Phi$.

Proof. As $n \mid k$, we can write $k = an$ with $a \in \mathbb{N}$. By 6.3 we get that Φ is of the form

$$\Phi = (x, \begin{pmatrix} x & 0 \\ x - \frac{ny}{m} & \frac{ny}{m} \end{pmatrix} + d \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$$

where we may assume $0 \le y < m$ and $0 \le d < n$, and where $\frac{m}{(n,m)} \mid y$.

As $(a, 0, a) \in \mathrm{K}_0(\mathbb{I}_n^\sim, \mathbb{Z} \oplus \mathbb{Z}/k)^+$, we conclude that the image $(xa, da, a(\frac{ny}{m} + d))$ is positive. Since $0 \le d < n$, we see that $0 \le ad < an = k$, which means that ad is the minimal representative and we may conclude that $ad \le ax$, hence $d \le x$. Now, $a(\frac{ny}{m} + d)$ may not be the minimal representative, as we only know that $0 \le a(\frac{ny}{m} + d) < an + an = 2k$. We consider the cases $0 \le a(\frac{ny}{m} + d) < k$ and $k \le a(\frac{ny}{m} + d) < 2k$ seperately.

If $0 \le a(\frac{ny}{m}+d) < k$, then $a(\frac{ny}{m}+d) \le ax$ hence $\frac{ny}{m}+d \le x$. As $\frac{m}{(n,m)} \mid y$ we can write $y = z\frac{m}{(n,m)}$ with $z \in \mathbb{N}_0$, and define

$$\tilde{\Phi} = \Phi - z\,\mathbb{K}(\mathrm{in}_{m,n};k) = (x - \frac{ny}{m}, \begin{pmatrix} x-\frac{ny}{m} & 0 \\ x-\frac{ny}{m} & 0 \end{pmatrix} + d\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, 0).$$

We want $\tilde{\Phi}$ to belong to $\mathrm{Hom}(\mathbb{K}(\mathbb{I}_n^\sim;k),\mathbb{K}(\mathbb{I}_m^\sim;k))^+$ as this will allow us to use 6.5. As $x-\frac{ny}{m} \ge d \ge 0$, the K_*-part of $\tilde{\Phi}$ is positive. The four generators of $\mathrm{K}_0(\mathbb{I}_n^\sim;\mathbb{Z}\oplus\mathbb{Z}/k)^+$ are $(1,0,0)$, $(1,1,1)$, $(a,0,a)$ and $(a,a,0)$ as $a = \frac{k}{(n,k)}$, and they are mapped to $(x-\frac{ny}{m},0,0)$, $(x-\frac{ny}{m},x-\frac{ny}{m},x-\frac{ny}{m})$, $a(x-\frac{ny}{m},d,d)$ and $a(x-\frac{ny}{m},x-\frac{ny}{m}-d,x-\frac{ny}{m}-d)$ which therefore are the four elements we desire to be positive. The two first elements are clearly positive, and as $x \ge \frac{ny}{m}+d$ gives us that $a(x-\frac{ny}{m}) \ge ad \ge 0$ and $a(x-\frac{ny}{m}) \ge a(x-\frac{ny}{m}-d) \ge 0$, the two last elements are positive as well. Therefore $\tilde{\Phi}$ is positive and 6.5 provides us with $a_0,a_1 \in \mathbb{N}_0$ such that $\Gamma^k(a_0[\eta\delta_0]+a_1[\eta\delta_1]) = \tilde{\Phi}$, and by setting $a_2 = z$ and $a_3 = 0$ we obtain the desired as $\Phi = \tilde{\Phi} + z\,\mathbb{K}(\mathrm{in}_{m,n};k)$.

If $k \le a(\frac{nu}{m}+d) < 2k$, then $a(\frac{ny}{m}+d)-k$ is the minimal representative, hence $0 \le \frac{ny}{m}+d-n < n$, and $\frac{ny}{m}+d-n < x$ as then $a(\frac{ny}{m}-d)-k \le ax$. Write $m-y = z\frac{m}{(n,m)}$ with $z \in \mathbb{N}_0$, and define

$$\tilde{\Phi} = \Phi - z\,\mathbb{K}(\overline{\mathrm{in}}_{m,n};k) = (x-n+\frac{ny}{m}, \begin{pmatrix} x & \frac{ny}{m}-n \\ x-n & \frac{ny}{m} \end{pmatrix} + d\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, 0).$$

Again we desire $\tilde{\Phi}$ to be positive. The K_*-part is positive as $x-n+\frac{ny}{m} \ge d-n+\frac{ny}{m} \ge 0$. The four generators of $\mathrm{K}_0(\mathbb{I}_n^\sim;\mathbb{Z}\oplus\mathbb{Z}/k)^+$ are mapped to $(x-n+\frac{ny}{m},0,0)$, $(x-n+\frac{ny}{m},x-n+\frac{ny}{m},x-n+\frac{ny}{m})$, $a(x-n+\frac{ny}{m},\frac{ny}{m}-n+d,\frac{ny}{m}+d)$ and $a(x-n+\frac{ny}{m},x-d,x-n-d)$. Again the two first elements are clearly positive, the third element is positive since $an = k$ and $0 \le a(\frac{ny}{m}-n+d) \le a(x-n+\frac{ny}{m})$ as $d \le x$, and the fourth element is positive since $an = k$ and $0 \le a(x-d) \le a(x-n+\frac{ny}{m})$ as $0 \le d-n+\frac{ny}{m}$. Hence $\tilde{\Phi}$ is positive and 6.5 provides us with $a_0,a_1 \in \mathbb{N}_0$ such that $\tilde{\Phi} = \Gamma^k(a_0[\eta\delta_0]+a_1[\eta\delta_1])$, and by setting $a_2 = 0$ and $a_3 = z$ we obtain the desired. ♡

The lifting result for finite direct sums of matrices over building blocks can now be achieved. It is one of the key tools in the proof of the main theorem.

Proposition 6.7 Let A and B be finite direct sums of matrix algebras over building blocks. If N exceeds the maximal size m of dimension drop algebras $\mathbb{I}_m^\sim$ occuring in A, then

$$\Gamma^n(\mathrm{KK}(A,B)^{+,\Sigma}) \supseteq \mathrm{Hom}(\mathbb{K}(A;n),\mathbb{K}(B;n))^{+,\Sigma,N}.$$

Proof. Firstly, we consider the case where $A = M_k(C)$ and $B = M_l(D)$ are matrices over building blocks C and D. Let $\Phi = (\varphi_0,\varphi,\varphi_1) \in \mathrm{Hom}(\mathbb{K}(A;n),\mathbb{K}(B;n))^{+,\Sigma,N}$ be given. Notice that φ_0 is multiplication by some $x \in \mathbb{Z}$ since $\mathrm{K}_0(A) = K_0(B) = \mathbb{Z}$, that $x \ge 0$ since φ_0 is positive and $\mathrm{K}_0(A)^+ = \mathrm{K}_0(B)^+ = \mathbb{N}_0$, and that $xk \le l$ since φ_0 respects the scales and $\Sigma(A) = \{0,\dots,k\}$ and $\Sigma(B) = \{0,\dots,l\}$. So, if we can find a $*$-homomorphism $\alpha\colon C \to M_x(D)$ such that $\mathbb{K}(\alpha;n) = \Phi$, we can define

$\tilde{\alpha} = \mathrm{diag}(\alpha \otimes 1_k, 0, \dots, 0)\colon M_k(C) \to M_l(D)$ as $xk \leq l$, and the desired will follow as $[\tilde{\alpha}] \in \mathrm{KK}(A,B)^{+,\Sigma}$ and $\mathbb{K}(\tilde{\alpha};n) = \mathbb{K}(\alpha;n) = \Phi$.

The construction of $\alpha\colon C \to M_x(D)$ depends on what kinds of building blocks C and D are. We consider first the case where both $C = \mathbb{I}_m^{\sim}$ and $D = \mathbb{I}_p^{\sim}$ are dimension drop algebras. As by stability $\Phi \in \mathrm{Hom}(\mathbb{K}(C;n), \mathbb{K}(D;n))^{+,N}$, we get by 6.3 that

$$\Phi = (x, \begin{pmatrix} x & 0 \\ x - \frac{my}{p} & \frac{my}{p} \end{pmatrix} + d \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$$

for some $d, y \in \mathbb{Z}$ with $0 \leq d < (n,m)$ and $0 \leq y < p$. We consider the two cases $y = 0$ and $y \neq 0$ separately. If $y = 0$, then 6.5 gives us $a_0, a_1 \in \mathbb{N}_0$ such that $\Gamma^n(a_0[\eta\delta_0] + a_1[\eta\delta_1]) = \Phi$ and thereby in particular $x = a_0 + a_1$, so we may define $\alpha = \mathrm{diag}(\eta\delta_0, \dots, \eta\delta_0, \eta\delta_1, \dots, \eta\delta_1)\colon C \to M_x(D)$ to consist of first a_0 copies of $\eta\delta_0$ and then a_1 copies of $\eta\delta_1$, and clearly $\mathbb{K}(\alpha;n) = \Phi$. If $y \neq 0$, then $x \geq N \geq m$ as (φ_0, φ_1) is N-large. We notice now that in this case

$$(x, \begin{pmatrix} x & 0 \\ x - \frac{my}{p} & \frac{my}{p} \end{pmatrix} + d \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y) \in \mathrm{Hom}(\mathbb{K}(C;m), \mathbb{K}(D;m))^+$$

as the generators $(1,0,0)$, $(1,1,1)$, $(1,0,1)$ and $(1,1,0)$ of $\mathrm{K}_0(C; \mathbb{Z} \oplus \mathbb{Z}/m)^+$ are mapped to $(x,0,0)$, (x,x,x), $(x,0,\frac{my}{p})$ and $(x,x,x-\frac{my}{p})$ which are all positive as $x \geq m \geq \frac{my}{p} \geq 0$. We may therefore conclude by 6.6 together with 6.5 that there is $a_0, a_1, a_2, a_3 \in \mathbb{N}_0$ and $0 \leq d' < m$ such that

$$\Gamma^q(a_0[\eta\delta_0] + a_1[\eta\delta_1] + a_2[\mathrm{in}_{p,m}] + a_3[\overline{\mathrm{in}}_{p,m}]) = (x, \begin{pmatrix} x & 0 \\ x - \frac{my}{p} & \frac{my}{p} \end{pmatrix} + d' \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$$

for all q and where $d' \equiv d \pmod{(m,m)}$ as

$$\Gamma^m(a_0[\eta\delta_0] + a_1[\eta\delta_1] + a_2[\mathrm{in}_{p,m}] + a_3[\overline{\mathrm{in}}_{p,m}]) = (x, \begin{pmatrix} x & 0 \\ x - \frac{my}{p} & \frac{my}{p} \end{pmatrix} + d \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y).$$

Notice that $x = a_0 \mathrm{K}_0(\eta\delta_0) + a_1 \mathrm{K}_0(\eta\delta_1) + a_2 \mathrm{K}_0(\mathrm{in}_{p,m}) + a_3 \mathrm{K}_0(\overline{\mathrm{in}}_{p,m}) = a_0 + a_1 + \frac{m}{(m,p)}(a_2 + a_3)$, hence as $\mathrm{in}_{p,m}, \overline{\mathrm{in}}_{p,m}\colon C \to M_{\frac{m}{(m,p)}}(D)$ we may define our $*$-homomorphism as $\alpha = \mathrm{diag}(\eta\delta_0, \dots, \eta\delta_0, \eta\delta_1, \dots, \eta\delta_1, \mathrm{in}_{p,m}, \dots, \mathrm{in}_{p,m}, \overline{\mathrm{in}}_{p,m}, \dots, \overline{\mathrm{in}}_{p,m})\colon C \to M_x(D)$ consisting of a_0 copies of $\eta\delta_0$, a_1 of $\eta\delta_1$, a_2 of $\mathrm{in}_{p,m}$, and a_3 of $\overline{\mathrm{in}}_{p,m}$. As $d' \equiv d \pmod{(n,m)}$, we see that $\mathbb{K}(\alpha;n) = \Gamma^n([\alpha]) = \Phi$.

If $C = \mathbb{I}_m^{\sim}$ and $D = C(S^1)$, then similarly 6.5 gives us $a_0, a_1 \in \mathbb{N}_0$ such that $\Gamma^n(a_0[\eta\delta_0] + a_1[\eta\delta_1]) = \Phi$, hence $a_0 + a_1 = x$ and we define $\alpha = \mathrm{diag}(\eta\delta_0, \dots, \eta\delta_0, \eta\delta_1, \dots, \eta\delta_1)\colon C \to M_x(D)$ and obtain $\mathbb{K}(\alpha;n) = \Phi$.

If $C = C(S^1)$, then the proof of 6.2 tells us that Φ is dictated by (φ_0, φ_1), hence we need only find $\alpha\colon C \to M_x(D)$ satisfying $\mathrm{K}_*(\alpha) = (\varphi_0, \varphi_1)$. Write $(\varphi_0, \varphi_1) = (x, y)$, and notice that since (φ_0, φ_1) is N-large, $x = 0$ will imply $y = 0$ in which case $\alpha = 0$ will do, hence we may assume that $x \geq 1$. Consider the element $u \in C(S^1)$ given by $u(t) = t$, and notice that u generates $C(S^1)$ as a C^*-algebra, hence we may define a unital

$*$-homomorphism $\alpha\colon C \to M_x(D)$ merely be defining $\alpha(u)$, as long as $\alpha(u)$ is set to be a unitary element. In the case where $D = C(S^1)$, we define $\alpha(u) = \mathrm{diag}(u^y, 1, \dots, 1)$. And in the case where $D = \mathbb{I}_p^\sim$, we define $\alpha(u) = \mathrm{diag}(u_p^y, 1, \dots, 1)$, where $u_p \in \mathbb{I}_p^\sim$ is defined as $u_p(t) = \mathrm{diag}(e^{2\pi i t}, 1, \dots, 1)$. As 1 generate $\mathrm{K}_0(C(S^1))$ and $\mathrm{K}_0(\mathbb{I}_p^\sim)$, we see that $\mathrm{K}_0(\alpha) = x$ in both cases, and as u generate $\mathrm{K}_1(C(S^1))$ and u_p generate $\mathrm{K}_1(\mathbb{I}_p^\sim)$, we see that $\mathrm{K}_1(\alpha) = y$. Hence we have obtained $\mathbb{K}(\alpha; n) = \Phi$ in both cases.

Secondly, in the general case we have $A = A_1 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus \cdots B_k$ being finite direct sums of matrices over building blocks. Consider the canonical injections and projections $\iota_i^A\colon A_i \to A$, $\iota_j^B\colon B_j \to B$, $\pi_i^A\colon A \to A_i$ and $\pi_j^B\colon B \to B_j$, and notice how for any $\Phi \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))^{+,\Sigma,N}$ we have $\mathbb{K}(\pi_j^B; n)\Phi\,\mathbb{K}(\iota_i^A; n) \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))^{+,\Sigma,N}$. By the above shown we may lift to $*$-homomorphisms $\alpha_{ji}\colon A_i \to B_j$ satisfying $\mathbb{K}(\alpha_{ji}; n) = \mathbb{K}(\pi_j^B; n)\Phi\,\mathbb{K}(\iota_i^A; n)$, and by defining $\alpha\colon A \to B$ as $\alpha = \sum \iota_j^B \alpha_{ji} \pi_i^A$ we achieve $\mathbb{K}(\alpha; n) = \Phi$, cf. 2.9. ♡

6.2 Completeness of $\mathbb{K}(-; n)$

As mentioned, the main result is that any isomorphism between $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$ may be lifted to an isomorphism between A and B, when the AD algebras A and B satisfy some demands. One of the demands being that $n \operatorname{tor} \mathrm{K}_1(-) = 0$, which means that β_n is surjective on the torsion part of $\mathrm{K}_1(-)$ and therefore in a sense keeps track of it. The proof is from [Eil95], and but a few details have been added, in particular when dealing with the claim that $n \operatorname{tor} \operatorname{im} \mathrm{K}_1(f_i) = 0$ may be assumed.

The strategy of the proof is quite standard, namely to pull back the isomorphism of the K-groups to homomorphisms of the K-groups of the finite direct sums of matrices over building blocks, and then lift these to a KK-shape equivalence that will induce the desired isomorphism of the C^*-algebras.

Theorem 6.8 The invariant $\mathbb{K}(-; n)$ is strongly complete for the class of real rank zero AD algebras with $n \operatorname{tor} \mathrm{K}_1(-) = 0$.

Proof. Let real rank zero AD algebras A and B be given, and assume that $n \operatorname{tor} \mathrm{K}_1(A) = 0$, $n \operatorname{tor} \mathrm{K}_1(B) = 0$, and that $\Phi = (\varphi_0, \varphi, \varphi_1) \in \mathrm{Hom}(\mathbb{K}(A;n), \mathbb{K}(B;n))$ an isomorphishim is given. Let inductive systems (A_i, f_i) and (B_i, g_i) of matrix algebras over building blocks be given, and assume that $A = \varinjlim(A_i, f_i)$ and $B = \varinjlim(B_i, g_i)$.

We wish to invoke theorem 3.12 to prove the existence of an isomorphism $\alpha\colon A \to B$ satisfying $\mathbb{K}(\alpha; n) = \Phi$, and we therefore construct a suitable KK-shape equivalence of subsystems of (A_i, f_i) and (B_i, g_i).

Recall that the groups $\mathrm{K}_0(A_i)$, $\mathrm{K}_0(A_i; \mathbb{Z}/n)$ and $\mathrm{K}_1(A_i)$ are finite direct sums of cyclic groups, cf. 4.9 and 4.7. Recall also that by 4.15, $\mathbb{K}(-; n)$ is continuous.

First of all we notice that we can assume that

$$n \operatorname{tor} \operatorname{im} \mathrm{K}_1(f_i) = 0 \quad \text{and} \quad n \operatorname{tor} \operatorname{im} \mathrm{K}_1(g_i) = 0$$

for any i. To see this, consider the subgroup $G = \{x \in \mathrm{K}_1(A_i) \mid \mathrm{K}_1(f_{\infty,i})(x) \in \operatorname{tor} \mathrm{K}_1(A_i)\}$ of $\mathrm{K}_1(A_i)$. Recall that any submodule of a finitely generated $\mathbb{Z}$-module is finitely generated, as $\mathbb{Z}$ is a principal ideal domain. So as $\mathrm{K}_1(A_i)$ is finitely generated, so is G, and we let $x_1, \dots, x_m$ denote a set of generators for G. For each x_μ we have $\mathrm{K}_1(f_{\infty,i})(x_\mu) \in \operatorname{tor}\mathrm{K}_1(A)$ and thereby $n\,\mathrm{K}_1(f_{\infty,i})(x_\mu) = 0$, hence we can find a $j_\mu \geq i$ such that $n\,\mathrm{K}_1(f_{j,i})(x_\mu) = 0$ whenever $j \geq j_\mu$. Put $j = \max\{j_1, \dots, j_m\}$, and the claim is that $n \operatorname{tor}\operatorname{im}\mathrm{K}_1(f_{j,i}) = 0$. For any $x \in \mathrm{K}_1(A_i)$ we see that if $\mathrm{K}_1(f_{j,i})(x) \in \operatorname{tor}\mathrm{K}_1(A_j)$ then x lies in G and is therefore on the form $x = n_1x_1 + \cdots + n_mx_m$ whereby $n\,\mathrm{K}_1(f_{j,i})(x) = 0$ follows as $n\,\mathrm{K}_1(f_{j,i})(x_\mu) = 0$. Now, as we for any i can find a $j \geq i$ such that $n \operatorname{tor}\operatorname{im}\mathrm{K}_1(f_{j,i})$, we see that by discarding some A_is and renumbering, we may assume $n \operatorname{tor}\operatorname{im}\mathrm{K}_1(f_i) = 0$ hold for every i. Similarly we may assume $n \operatorname{tor}\operatorname{im}\mathrm{K}_1(g_i) = 0$.

We know describe how one for each i can construct a $j \geq i$ and a $\Psi = (\psi_0, \psi, \psi_1) \in \operatorname{Hom}(\mathbb{K}(A_i; n), \mathbb{K}(B_j; n))^{+,\Sigma,N}$ satisfying $\mathbb{K}(g_{\infty,j}; n)\Psi = \Phi\,\mathbb{K}(f_{\infty,i}; n)$ and where N denotes the maximal size m of dimension drop algebras $\mathbb{I}_m^\sim$ occuring in A_i.

We begin by constructing a homomorphism $\psi\colon \mathrm{K}_0(A_i; \mathbb{Z}/n) \to \mathrm{K}_0(B_j; \mathbb{Z}/n)$ satisfying $\mathrm{K}_0(g_{\infty,j}; \mathbb{Z}/n)\psi = \varphi\,\mathrm{K}_0(f_{\infty,i}; \mathbb{Z}/n)$. Let $x_1, \dots, x_m$ denote generators for the m summands of $\mathrm{K}_0(A_i; \mathbb{Z}/n)$. For each x_μ we can find a j_μ such that $\varphi(\mathrm{K}_0(f_{\infty,i})(x_\mu)) \in \operatorname{im}\mathrm{K}_0(g_{\infty,j_\mu})$ and thereby a $y_\mu \in \mathrm{K}_0(B_{j_\mu}; \mathbb{Z}/n)$ with $\varphi(\mathrm{K}_0(f_{\infty,i})(x_\mu)) = \mathrm{K}_0(g_{\infty,j_\mu})(y_\mu)$, and if x_μ is a torsion element of order k then since $\varphi(\mathrm{K}_0(f_{\infty,i})(kx_\mu)) = 0$ we can obtain $ky_\mu = 0$ by choosing j_μ sufficiently large. Putting $j = \max\{j_1, \dots, j_m\}$ we can now define a homomorphism $\psi\colon \mathrm{K}_0(A_i; \mathbb{Z}/n) \to \mathrm{K}_0(B_j; \mathbb{Z}/n)$ by $x_\mu \mapsto \mathrm{K}_0(g_{j,j_\mu}; \mathbb{Z}/n)(y_\mu)$ and expanding by $\mathbb{Z}$-linearity. This j will probably not be large enough to satisfy our further demands, and each time we replace j by a larger j' it is implied that ψ is replaced by $\mathrm{K}_0(g_{j',j}; \mathbb{Z}/n)\psi$.

The maps $\psi_0\colon \mathrm{K}_0(A_i) \to \mathrm{K}_0(B_j)$ and $\psi_1\colon \mathrm{K}_1(A_i) \to \mathrm{K}_1(B_j)$ are constructed similarly by replacing j with a number sufficiently larger. We wish for (ψ_0, ψ) and (ψ_0, ψ_1) to be positive, and as $\mathrm{K}_0(A_i; \mathbb{Z}/n)^+$ is finitely generated, as (φ_0, φ) is positive, and as $\mathrm{K}_0(B; \mathbb{Z} \oplus \mathbb{Z}/n)^+ = \varinjlim \mathrm{K}_0(B_k; \mathbb{Z} \oplus \mathbb{Z}/n)^+$, we see that (ψ_0, ψ) will be positive if we choose j sufficiently large. Since $\mathrm{K}_*(C(S^1))^+$ isn't finitely generated as $\mathrm{K}_*(\mathbb{I}_m^\sim)^+$ is, it seems harder to make (ψ_0, ψ_1) positive. But notice that if (x, y) is a positive element of $\mathrm{K}_*(C(S^1))$ or $\mathrm{K}_*(\mathbb{I}_m^\sim)$, then (x, ky) is as well for any $k \in \mathbb{Z}$. As any positive element of $\mathrm{K}_*(C(S^1))$ can be written as a finite sum of elements of the form $(1, k)$, we conclude that (ψ_0, ψ_1) will be positive on a summand of the form $\mathrm{K}_*(C(S^1))$ if it is positive on the summand's element $(1, 1)$. We can thereby make also (ψ_0, ψ_1) – and thereby the entire triplet (ψ_0, ψ, ψ_1) – positive merely by choosing j large enough.

And as $\Sigma(A_i)$ is finitely generated and $\Sigma(B) = \varinjlim \Sigma(B_k)$, ψ_0 will preserve the scale if we choose j large enough.

To obtain $\rho_n^{B_j}\psi_0 = \psi\rho_n^{A_i}$ and $\beta_n^{B_j}\psi = \psi_1\beta_n^{A_i}$, we only need to achieve it on the finitely many generators, and as

$$\mathrm{K}_0(g_{\infty,j}; \mathbb{Z}/n)\rho_n^{B_j}\psi_0 = \rho_n^B\varphi_0\,\mathrm{K}_0(f_{\infty,i}) = \varphi\rho_n^A\,\mathrm{K}_0(f_{\infty,i}) = \mathrm{K}_0(g_{\infty,j}; \mathbb{Z}/n)\psi\rho_n^{A_i}$$

and $\mathrm{K}_1(g_{\infty,j})\beta_n^{B_j}\psi = \mathrm{K}_1(g_{\infty,j})\psi_1\beta_n^{A_i}$, we only need to replace j by a step sufficiently larger.

As we want (ψ_0, ψ_1) to be N-large, we let lemma 3.8 provide us with a j' such that $\mathrm{K}_*(g_{j',j})$ is N-large. Being positive the maps (ψ_0, ψ_1) and $\mathrm{K}_*(g_{j',j})$ will be of the form $((x_{rs}),(y_{rs}))$ where $x_{rs} \geq 0$ and $x_{rs} > 0$ when $y_{rs} \neq 0$. Using this one can show by matrix multiplication that $\mathrm{K}_*(g_{j',j})(\psi_0,\psi_1)$ is N-large, and by replacing j with j' we obtain $\Psi = (\psi_0,\psi,\psi_1) \in \mathrm{Hom}(\mathbb{K}(A_i;n),\mathbb{K}(B_j;n))^{+,\Sigma,N}$. By a similar argument we notice that with this new j the map $\mathrm{K}_*(g_{j',j})(\psi_0,\psi_1)$ is N-large for any $j' \geq j$.

Using the same procedure we can construct a $\tilde{\Psi} \in \mathrm{Hom}(\mathbb{K}(B_{j+1};n),\mathbb{K}(A_k;n))^{+,\Sigma,M}$ with $\mathbb{K}(f_{\infty,k};n)\tilde{\Psi} = \Phi^{-1}\,\mathbb{K}(g_{\infty,j+1};n)$ where M denotes the maximal size m of dimension drop algebras $\mathbb{I}_m^{\sim}$ occuring in B_{j+1}. And as

$$\mathbb{K}(f_{\infty,i};n) = \Phi^{-1}\,\mathbb{K}(g_{\infty,j};n)\Psi = \mathbb{K}(f_{\infty,k};n)\tilde{\Psi}\,\mathbb{K}(g_j;n)\Psi$$

we can, by choosing a larger k, assume that $\mathbb{K}(f_{k,i};n) = \tilde{\Psi}\,\mathbb{K}(g_j;n)\Psi$.

Now, by using this construction recursively, discarding A_is and B_js and renumbering as we proceed, we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
\mathbb{K}(A_1;n) & \xrightarrow{\mathbb{K}(f_1;n)} & \mathbb{K}(A_2;n) & \xrightarrow{\mathbb{K}(f_2;n)} & \mathbb{K}(A_3;n) & \xrightarrow{\mathbb{K}(f_3;n)} & \mathbb{K}(A_4;n) & \xrightarrow{\mathbb{K}(f_4;n)} & \cdots & \xrightarrow{\mathbb{K}(f_{\infty,i};n)} & \mathbb{K}(A;n) \\
 & & \downarrow{\scriptstyle \Psi_1} & & \uparrow{\scriptstyle \tilde{\Psi}_1} & & \downarrow{\scriptstyle \Psi_2} & & & & {\scriptstyle \Phi^{-1}}\uparrow\downarrow{\scriptstyle \Phi} \\
\mathbb{K}(B_1;n) & \xrightarrow{\mathbb{K}(g_1;n)} & \mathbb{K}(B_2;n) & \xrightarrow{\mathbb{K}(g_2;n)} & \mathbb{K}(B_3;n) & \xrightarrow{\mathbb{K}(g_3;n)} & \mathbb{K}(B_4;n) & \xrightarrow{\mathbb{K}(g_4;n)} & \cdots & \xrightarrow{\mathbb{K}(g_{\infty,i};n)} & \mathbb{K}(B;n)
\end{array}$$

with homomorphisms $\Psi_i \in \mathrm{Hom}(\mathbb{K}(A_{2i};n),\mathbb{K}(B_{2i};n))^{+,\Sigma,N_i}$ and homomorphisms $\tilde{\Psi}_i \in \mathrm{Hom}(\mathbb{K}(B_{2i+1};n),\mathbb{K}(A_{2i+1};n))^{+,\Sigma,M_i}$ where N_i and M_i denote the maximal size m of dimension drop algebras $\mathbb{I}_m^{\sim}$ occuring in A_{2i} respectively B_{2i+1}.

Proposition 6.7 gives us homomorphisms $\psi_i\colon A_{2i} \to B_{2i}$ and $\tilde{\psi_i}\colon B_{2i+1} \to A_{2i+1}$ satisfying $\mathbb{K}(\psi_i;n) = \Psi_i$ and $\mathbb{K}(\tilde{\psi_i};n) = \tilde{\Psi}_i$. We want to realize that the diagram

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_{3,1}} & A_3 & \xrightarrow{f_{5,3}} & A_5 & \xrightarrow{f_{7,5}} & A_7 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \psi_1 f_1} & \nearrow{\scriptstyle \tilde{\psi}_1 g_2} & \downarrow{\scriptstyle \psi_2 f_3} & \nearrow{\scriptstyle \tilde{\psi}_2 g_4} & \downarrow{\scriptstyle \psi_3 f_5} & \nearrow{\scriptstyle \tilde{\psi}_3 g_6} & \downarrow{\scriptstyle \psi_4 f_7} & \nearrow & \\
B_2 & \xrightarrow[g_{4,2}]{} & B_4 & \xrightarrow[g_{6,4}]{} & B_6 & \xrightarrow[g_{8,6}]{} & B_8 & \longrightarrow & \cdots
\end{array}$$

commutes at the level of KK-theory. To make the part

$$\begin{array}{ccccc}
A_{2i-1} & \xrightarrow{f_{2i-1}} & A_{2i} & \xrightarrow{f_{2i}} & A_{2i+1} \\
 & & \downarrow{\scriptstyle \psi_i} & & \uparrow{\scriptstyle \tilde{\psi}_i} \\
 & & B_{2i} & \xrightarrow{g_{2i}} & B_{2+1}.
\end{array}$$

commute at the level of KK-theory it sufficies to prove that it does when A_{2i-1}, A_{2i}, and B_{2i+1} are building blocks, cf. remark 2.9. If both A_{2i-1} and A_{2i} are dimension drop algebras, then the desired follows from lemma 6.9. Elsewise if e.g. A_{2i-1} is a circle algebra then $K_*(A_{2i-1})$ is free and the desired follows from the injectivity stated in 6.2. Similarly one sees that the part

$$\begin{array}{ccccc}
 & & A_{2i+1} & \xrightarrow{f_{2i+1}} & A_{2i+2} \\
 & & \uparrow{\scriptstyle \tilde{\psi}_i} & & \downarrow{\scriptstyle \psi_{i+1}} \\
B_{2i} & \xrightarrow{g_{2i}} & B_{2i+1} & \xrightarrow{g_{2i+1}} & B_{2i+2}
\end{array}$$

commutes at the level of KK-theory.

By theorem 3.12 we get an isomorphism $\alpha\colon A \to B$ that satisfy

$$[\alpha f_{\infty,2i-1}] = [g_{\infty,2i}\psi_i f_{2i-1}] \quad \text{and} \quad [\alpha^{-1} g_{\infty,2i}] = [f_{\infty,2i+1}\tilde{\psi}_i g_{2i}]$$

and thereby in particular

$$\mathbb{K}(\alpha; n)\,\mathbb{K}(f_{\infty,2i-1}; n) = \mathbb{K}(g_{\infty,2i}; n)\Psi_i\,\mathbb{K}(f_{2i-1}; n) = \Phi\,\mathbb{K}(f_{\infty,2i-1}; n)$$

from which $\mathbb{K}(\alpha; n) = \Phi$ follows. ♡

The following lemma was needed in the proof of 6.8.

Lemma 6.9 Consider maps

$$\mathbb{I}_m^{\sim} \xrightarrow{\varphi} \mathbb{I}_k^{\sim} \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} A$$

where A is a building block. If $n\operatorname{tor}\operatorname{im}\mathrm{K}_1(\varphi) = 0$ and $\mathbb{K}(\psi_1; n) = \mathbb{K}(\psi_2; n)$, then $[\psi_1\varphi] = [\psi_2\varphi]$ as elements of $\mathrm{KK}(\mathbb{I}_m^{\sim}, A)$.

Proof. Since $\mathrm{K}_*(\psi_1) = \mathrm{K}_*(\psi_2)$, we only need prove $\mathrm{K}_0(\psi_1\varphi; \mathbb{Z}/m) = \mathrm{K}_0(\psi_2\varphi; \mathbb{Z}/m)$ to obtain first $\mathbb{K}(\psi_1\varphi; m) = \mathbb{K}(\psi_2\varphi; m)$ and then by the injectivity stated in 6.3 or 6.4 that $[\psi_1\varphi] = [\psi_2\varphi]$. Now, either A is a dimension drop algebra $\mathbb{I}_l^{\sim}$ or A is the circle algebra $C(S^1)$, and we handle the two cases separately; they are however quite similar.

Assume firstly that $A = \mathbb{I}_l^{\sim}$. According to 6.3, and as $\mathrm{K}_*(\psi_1) = \mathrm{K}_*(\psi_2)$, there exist $d_1, d_2 \in \mathbb{Z}$ and $x, y \in \mathbb{Z}$ with $0 \le d_i < k$ and such that

$$\mathbb{K}(\psi_i; r) = (x, \begin{pmatrix} x & 0 \\ x - \frac{ky}{l} & \frac{ky}{l} \end{pmatrix} + d_i \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, y)$$

for any $r \ge 2$. As for a fixed r, the number d_i is unique modulo (k, r), we conclude since $\mathbb{K}(\psi_1; n) = \mathbb{K}(\psi_2; n)$ that $(k, n) \mid d_1 - d_2$. Write $d_1 - d_2 = a(k, n)$. By 6.3 again, we get $z, w, e \in \mathbb{Z}$ with $0 \le e < m$ and satisfying

$$\mathbb{K}(\varphi; r) = (z, \begin{pmatrix} z & 0 \\ z - \frac{mw}{k} & \frac{mw}{k} \end{pmatrix} + d_i \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, w)$$

for all $r \ge 2$. As $n\operatorname{tor}\operatorname{im}\mathrm{K}_1(\varphi) = 0$ and $\mathrm{K}_1(\mathbb{I}_k^{\sim}) = \mathbb{Z}/k$, we notice that $k \mid nw$ whereof one may conclude that $k \mid (k, n)w$. By matrix multiplication we see that

$$\mathrm{K}_0(\psi_i\varphi; \mathbb{Z}/r) = \begin{pmatrix} xz & 0 \\ xz - \frac{myw}{l} & \frac{myw}{l} \end{pmatrix} + (ex + d_i\frac{mw}{k}) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

for all $r \ge 2$. Now,

$$(d_1 - d_2)\frac{mw}{k} = \frac{a(n,k)mw}{k} = \left(a\frac{(k,n)w}{k}\right)m,$$

so we conclude that $m \mid (e + d_1\frac{mw}{k}) - (e + d_2\frac{mw}{k})$ hence $\mathrm{K}_0(\psi_1\varphi; \mathbb{Z}/m) = \mathrm{K}_0(\psi_2\varphi; \mathbb{Z}/m)$.

Assume secondly that $A = C(S^1)$. According to 6.4, and as $\mathrm{K}_*(\psi_1) = \mathrm{K}_*(\psi_2)$, there exist $d_1, d_2 \in \mathbb{Z}$ and $x \in \mathbb{Z}$ with $0 \leq d_i < k$ and such that

$$\mathbb{K}(\psi_i; r) = (x, \begin{pmatrix} x & 0 \end{pmatrix} + d_i \begin{pmatrix} -1 & 1 \end{pmatrix}, 0)$$

for any $r \geq 2$. Again we conclude as $\mathbb{K}(\psi_1; n) = \mathbb{K}(\psi_2; n)$ that $(k, n) \mid d_1 - d_2$. As before we have by 6.3 that

$$\mathbb{K}(\varphi; r) = (z, \begin{pmatrix} z & 0 \\ z - \frac{mw}{k} & \frac{mw}{k} \end{pmatrix} + d_i \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, w)$$

for all $r \geq 2$, and we recall that $k \mid (k, n)w$. By matrix multiplication we see that

$$\mathrm{K}_0(\psi_i \varphi; \mathbb{Z}/r) = \begin{pmatrix} xz & 0 \end{pmatrix} + (ex + d_i \frac{mw}{k}) \begin{pmatrix} -1 & 1 \end{pmatrix}$$

for all $r \geq 2$, and as before we may conclude that $m \mid (e + d_1 \frac{mw}{k}) - (e + d_2 \frac{mw}{k})$ hence $\mathrm{K}_0(\psi_1 \varphi; \mathbb{Z}/m) = \mathrm{K}_0(\psi_2 \varphi; \mathbb{Z}/m)$. ♡

Remark 6.10 In [Eil95, 6.3.2] it is shown that the invariant $\mathbb{K}(-;\infty)$, mentioned in 4.14, is strongly complete for the class of real rank zero AD algebras, with no limitations on $\operatorname{tor} \mathrm{K}_1(-)$. The strategy of proof is quite similar to that of the completeness of $\mathbb{K}(-;n)$; namely that a lifting result like 6.7 is established and used to make an isomorphism of $\mathbb{K}(A;\infty)$ and $\mathbb{K}(B;\infty)$ into a KK-shape equivalence of A and B. As with 6.7 and 6.8, there is a lot of technical work in it, however.

7 Exploiting a range result for the invariant

A range result for an invariant F is a description of exactly which objects that occur as $F(A)$ for some A. Or rather, this is a range result for the objects, as one may also ask e.g. which automorphisms that occur as $F(\alpha)$ when $\alpha\colon A \to A$ runs through the automorphisms of A.

Now, in [ET] such a range result for the objects was made for the invariant $\mathbb{K}(-;n)$ (with the scale $\Sigma(-)$ omitted) and the class of real rank zero AD algebras, and in this section we will see how one directly from this range result can conclude that the invariant may be reduced if one adds some restrictions to the AD algebras, for example requires them to be simple. It is also mentioned how one may use the range result to prove irreducibility of the invariant.

Most importantly, in a way, we will investigate how the invariant carries more information than ordinary K-theory does. Because, as we will see, our invariant splits, so $\mathrm{K}_0(-;\mathbb{Z}/n)$ as a group is determined by the graded group $\mathrm{K}_*(-)$.

7.1 The range result

To just phrase the range result, we will be needing some terminology on ordered abelian groups from [Goo86] and [ET]. It will be introduced right after the theorem has been stated. We will not prove the theorem but merely import it. However, the entire section will rely on it.

By comparing the theorem to 6.8, one gets a bijection between real rank zero AD algebras A with $n \operatorname{tor} \mathrm{K}_1(A) = 0$ and so-called n-coefficient complexes $\mathbb{G}$ with $n \operatorname{tor} G_1 = 0$. Thus, we now have a completely algebraic description of the class of real rank zero AD algebras with bounded torsion in $\mathrm{K}_1(-)$.

Theorem 7.1 [ET, 5.4] Up to isomorphism, the n-coeffiecient complexes are exactly those that arise as $\mathbb{K}(A;n)$ (with the scale $\Sigma(A)$ omitted) where A is a real rank zero AD algebra.

Firstly, recall that an ordered group G is said to be *unperforated* if for all $x \in G$ and $n \in \mathbb{N}$, $nx \geq 0$ implies $x \geq 0$. Only torsion-free groups are unperforated. The group is called *weakly unperforated* if it just satisfies the following two conditions: for all $x \in G$ and $n \in \mathbb{N}$, $nx \geq 0$ implies that there exists $t \in G[n]$ such that $x + t \geq 0$; and, for all $x \in G^+$, $n \in \mathbb{N}$ and $t \in \operatorname{tor} G$, $nx+t \geq 0$ implies $x \pm t \geq 0$. Here $G[n] = \{g \in G \mid ng = 0\}$.

Notice that the latter condition is automatically satisfied if the group is a direct sum $G = G_0 \oplus G_1$ of ordered groups where the first group G_0 is torsion-free and where the first group G_0 *dominates* the order of G in the sense that $(x, y_1 \pm y_2) \geq 0$ whenever $(x, y_1), (x, y_2) \geq 0$. We will refer to such an ordered group $G_0 \oplus G_1$ where G_0 dominates the order as a *graded ordered group* – not to be confused with an ordered graded group. Clearly we obtain G_0 as an ordered group when we restrict the order of $G_0 \oplus G_1$ to $G_0 \oplus 0$, and we will often do so.

When considering an ordered group G we define an order on $G \oplus G$, making it into a graded ordered group, by saying that $(x_1, x_2) \geq 0$ when $x_1 \geq 0$ and $x_2 \in \{x \in G \mid \exists n \in$

$\mathbb{N} : -nx_1 \leq x \leq nx_1\}$. The subset $\{x \in G \mid \exists n \in \mathbb{N} : -nx_1 \leq x \leq nx_1\}$ is known as the order ideal generated by x_1. Also, a surjective group homomorphism $\varphi\colon G \to H$ from an ordered group G induces the *quotient order* from G on H by $H^+ = \varphi(G^+)$.

The *Riesz interpolation property* is that for any $x_1, x_2, y_1, y_2 \in G^+$ that satisfy $x_1+x_2 = y_1 + y_2$, one can find $z_{ij} \in G^+$ such that $x_i = z_{i1} + z_{i2}$ and $y_j = z_{1j} + z_{2j}$.

Definition 7.2 An n-coefficient complex $\mathbb{G}$ is an exact sequence

$$G_0 \xrightarrow{n} G_0 \xrightarrow{\rho} G_n \xrightarrow{\beta} G_1 \xrightarrow{n} G_1$$

of abelian groups where

- G_n is a $\mathbb{Z}/n$-module,
- $G_0 \oplus G_1$ and $G_0 \oplus G_n$ are graded ordered groups that restrict to the same order on G_0,
- $G_0 \oplus G_1$ has the Riesz interpolation property,
- G_0 is unperforated and $G_0 \oplus G_1$ is weakly unperforated,
- $G_0 \oplus \operatorname{im} \rho$ has the quotient order comming from $G_0 \oplus G_0$,
- $G_0 \oplus \operatorname{im} \beta$ has the quotient order comming from $G_0 \oplus G_n$.

We consider two such n-coefficient complexes $\mathbb{G}$ and $\mathbb{H}$ isomorphic if there exist group isomorphisms φ_0, φ and φ_1 making the following diagram commutative

$$\begin{array}{ccccc} G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} \\ H_0 & \xrightarrow{\rho} & H_n & \xrightarrow{\beta} & H_1 \end{array}$$

and which satisfy that $\varphi_0 \oplus \varphi$ and $\varphi_0 \oplus \varphi_1$ are order isomorphism, i.e. are positive and have positive inverses.

Remark 7.3 One can of course define a category of n-coefficient complexes where the morphisms are triplets $(\varphi_0, \varphi, \varphi_1)$ making the above diagram commutative satisfying that $\varphi_0 \oplus \varphi$ and $\varphi_0 \oplus \varphi_1$ are positive. This category seems a bit weird and one probably ought to explore it thoroughly. For instance, if the map $\varphi_0 \oplus \varphi_1$ is an order isomorphism and $\varphi_0 \oplus \varphi$ is positive, then automatically the map $\varphi_0 \oplus \varphi$ is in fact an order isomorphism.

7.2 n-coefficient complexes always split

As we will see, any n-coeffiecient complex splits, a fact that will come most useful to us. To prove it, we will need to perform a certain amount of group theory. More precisely, we will be needing the following result whose proof is taken from [Fuc70, 27.5]. Recall that a subgroup H of an abelian group G is called *pure* in G if $nG \cap H = nH$ holds for all $n \in \mathbb{N}$.

A standard example of a pure subgroup is a direct summand. One easily verifies that being a pure subgroup is a transitive property; and when considering a homomorphism $\varphi\colon G_1 \to G_2$, one can show that if H is pure in G_1 and $H \supseteq \ker\varphi$, then $\varphi(H)$ is pure in $\operatorname{im}\varphi$.

Proposition 7.4 If H is a bounded pure subgroup of the group G, then H is a direct summand in G.

Proof. As H is bounded, $\{|h| \mid h \in H\}$ is bounded, and we may hence prove the claim by complete induction over $\max\{|h| \mid h \in H\}$. Since H is a torsion-group it is the direct sum of its p-components $H_p = \{h \in H \mid |h| = p^k\}$, p running through the prime numbers, cf. [Fuc70, 8.4]; and as H is bounded the p-components are direct sums of cyclic groups $\mathbb{Z}/p^k$ and there is a maximal order q^l of the elements of the p-components, cf. [Fuc70, 17.2]. Let H_1 denote the direct sum of those summands in H_q that are isomorphic to $\mathbb{Z}/q^l$, and let K denote the sum of the rest of the summands along with the other p-components so that $H = H_1 \oplus K$ and $\max\{|k| \mid k \in K\} < \max\{|h| \mid h \in H\}$.

Clearly H_1 is pure in H, hence H_1 is also pure in G as H is pure in G. As H_1 is pure in G and is the direct sum of cyclic groups $\mathbb{Z}/q^l$, it follows from lemma 7.5 that H_1 is a direct summand in G. Write $G = H_1 \oplus G_1$.

Consider $G \to G_1$ the projection to the second coordinate. Under this map, H is mapped to $K_1 = H \cap G_1$ as $H \supseteq H_1$. So as H is pure in G and contains the kernel H_1, K_1 is pure in G_1. And one sees that $H = H_1 \oplus K_1$. Since $H_1 \oplus K = H_1 \oplus K_1$, K and K_1 are isomorphic as the direct sums are inner. Hence $\max\{|k| \mid k \in K_1\} = \max\{|k| \mid k \in K\} < \max\{|h| \mid h \in H\}$, and by the induction hypothesis we conclude that K_1 is a direct summand in G_1. Since $H = H_1 \oplus K_1$ and $G = H_1 \oplus G_1$, we conclude that H is a direct summand in G. ♡

To prove the lemma from [Fuc70, 9.8-9&27.1], needed in the above proof, we will be needing the concept of an *H-high subgroup* of G, H itself denoting a subgroup of G. An H-high subgroup is the most obvious candidate for a subgroup K that satisfies $G = H \oplus K$. It is defined as a subgroup K of G that intersects H trivially and is maximal in the sense that if $K \lneq K' \leq G$ then $K' \cap H \neq 0$. Zorn's lemma ensures us that H-high subgroups always exist.

Lemma 7.5 Let H be a direct sum of cyclic groups $\mathbb{Z}/p^n$ where p is a prime number. If H is a pure subgroup in G then H is in fact a direct summand in G.

Proof. As H is pure in G, $p^nG \cap H = p^nH = 0$, so by standard use of Zorn's lemma one can find an H-high subgroup K that contains p^nG. We want to prove that $G/(H \oplus K)$ is trivial, and we do this by showing that it is both a torsion-group and torsion-free.

First we notice that $G/(H \oplus K)$ is a torsion-group. Let $g \in G$ and assume that $g \notin H \oplus K$. Since $g \notin K$, $\mathbb{Z}g + K \gneq K$, hence $(\mathbb{Z}g + K) \cap H \neq 0$. I.e. there is some $h \in H$ such that $h \neq 0$ and $h \in \mathbb{Z}g + K$; write $h = lg + k$. Since $H \cap K = 0$, $l \neq 0$; so as $lg = h - k \in H \oplus K$, $g + (H \oplus K)$ is a torsion element.

Now we notice that $G/(H \oplus K)$ is torsion-free. To see this, we prove that for all prime numbers q, $qg \in H \oplus K$ implies $g \in H \oplus K$ for any $g \in G$. This might take awhile. Write $qg = h + k$. If $q \neq p$, then we have since $H = \bigoplus \mathbb{Z}/p^n$ an $h' \in H$ such that $ph' = h$. If $q = p$, we get since $p^n g = p^{n-1}h = p^{n-1}g$ and $p^n G \leq K$ that $p^{n-1}h \in H \cap K = 0$, so as $H = \bigoplus \mathbb{Z}/p^n$ there is some $h' \in H$ such that $ph' = h$. Now, $qg = qh' + k$ hence $q(g - h') \in K$. We will see that this implies that $g - h' \in H \oplus K$, whereby $g \in H \oplus K$ follows. Assume that $g - h' \notin K$. Then since $\mathbb{Z}(g - h') + K \gneq K$, we have $(\mathbb{Z}(g-h') + K) \cap H \neq 0$ and can find some nonzero $h'' \in H$ that may be written $h'' = l(g - h') + k'$. Here $(l, q) = 1$ as elsewise $q \mid l$ whereof $l(g - h') \in K$ and then $h'' \in K$ follows but $h'' \neq 0$ and $H \cap K = 0$. Write $1 = n_1 l + n_2 q$ as $(l, q) = 1$, and realise that now $g - h' = (n_1 l + n_2 q)(g - h') = n_1 h'' - n_1 k' + n_2 q(g - h') \in H \oplus K$. ♡

Observation 7.6 Let D denote an H-high subgroup in G. Then $(H \oplus D)/D$ is an essential subgroup in G/D, and the short-exact sequence

$$0 \longrightarrow H \xrightarrow{h \mapsto h+D} G/D \xrightarrow{g+D \mapsto g+(H \oplus D)} G/(H \oplus D) \longrightarrow 0$$

splits if and only if $G = H \oplus D$.

Corollary 7.7 Considering any n-coefficient complex $\mathbb{G}$, the short-exact sequence

$$0 \longrightarrow G_0/n \xrightarrow{x+nG_0 \mapsto \rho(x)} G_n \xrightarrow{\beta|^{G_1[n]}} G_1[n] \longrightarrow 0$$

splits.

Proof. Clearly $n \operatorname{im} \rho = 0$ as e.g. $nG_n = 0$, hence $\operatorname{im} \rho$ is bounded, so we need only to show that $\operatorname{im} \rho$ is pure in G_n to conclude by 7.4 that $\operatorname{im} \rho$ is a direct summand in G_n whereby it follows that the sequence splits.

Let $m \in \mathbb{N}$ and let us show that $mG_n \cap \operatorname{im} \rho \subseteq m \operatorname{im} \rho$. Let $x \in G_n$ and assume that $mx \in \operatorname{im} \rho$. Write $mx = \rho(y)$ for some $y \in G_0$, and write $(n, m) = an + bm$ with $a, b \in \mathbb{Z}$. As $nx = 0$, we see that $(n, m)x = bmx = \rho(by)$, hence $\frac{n}{(n,m)} by \in \ker \rho = nG_0$. Now, write $\frac{n}{(n,m)} by = ny'$ for some $y' \in G_0$, and notice as $\operatorname{tor} G_0 = 0$ that this implies that $by = (n, m)y'$. Clearly, $mx = \frac{m}{(n,m)}(n, m)x = \frac{m}{(n,m)}\rho(by) = m\rho(y') \in m \operatorname{im} \rho$, as desired. ♡

Remark 7.8 That $\mathrm{K}_0(A; \mathbb{Z}/n)$ and $\mathrm{K}_0(A)/n \oplus \mathrm{K}_1(A)[n]$ are isomorphic as groups, was already known by the (unnatural) splitting of the UCT (2.11):

$$\mathrm{KK}(\mathbb{I}_n, A) = \mathrm{Ext}^1(\mathbb{Z}/n, K_0(A)) \oplus \mathrm{Hom}(\mathbb{Z}/n, \mathrm{K}_1(A)) = \mathrm{K}_0(A)/n \oplus \mathrm{K}_1(A)[n]$$

as $\mathrm{K}_*(\mathbb{I}_n) = 0 \oplus \mathbb{Z}/n$. And now, by 7.7 and 7.1, we conclude furthermore that the complex $\mathbb{K}(A; n)$ splits for all real rank zero AD algebras A. A result one could also have obtained by studying [Böd79] and [Böd80], however. One may then ask how $\mathrm{K}_*(A)$, none the less, carries less information than $\mathbb{K}(A; n)$ does. The answer is that such a splitting doesn't always respect the orders on $\mathrm{K}_0(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_*(A)$.

Observation 7.9 When considering a graded ordered group $G_0 \oplus G_1$ where G_0 is unperforated and $G_0 \oplus G_1$ is weakly unperforated and has the Riesz interpolation property, one can always construct an n-coefficient complex

$$G_0 \longrightarrow G_0/n \oplus G_1[n] \longrightarrow G_1$$

simply by considering the canonical splitting maps $x \mapsto (x + nG_0, 0)$ and $(x, y) \mapsto y$, and by equipping $G_0 \oplus (G_0/n \oplus G_1[n])$ with the following order: $(x_1, x_2 + nG_0, y) \geq 0$ when $(x_1, x_2 + nG_0) \geq 0$ and $(x_1, y) \geq 0$, the order on $G_0 \oplus G_0/n$ being the quotient order inherited from $G_0 \oplus G_0$.

We will be using this constructing quite a bit, so unless otherwise specified $G_0/n \oplus G_1[n]$ is implicitly equipped with the order defined above.

Definition 7.10 Given an n-coefficient complex $\mathbb{G}$, a splitting map $\sigma \colon G_1[n] \to G_n$ is called *positive* if the map $\mathrm{id} \oplus \sigma \colon G_0 \oplus G_1[n] \to G_0 \oplus G_n$ is positive.

The following lemma tells us exactly when the graded ordered group $G_0 \oplus G_1$ of a given n-coefficient complex $\mathbb{G}$ only occurs in one n-coefficient complex.

Lemma 7.11 When considering an n-coefficient complex $\mathbb{G}$ and a splitting

$$0 \longrightarrow G_0/n \xrightarrow{x + nG_0 \mapsto \rho(x)} G_n \underset{\sigma}{\overset{\beta|^{G_1[n]}}{\rightleftarrows}} G_1[n] \longrightarrow 0,$$

and then defining a map $\varphi \colon G_0/n \oplus G_1[n] \to G_n$ as $(x, y) \mapsto \rho(x) + \sigma(y)$, then $\mathrm{id} \oplus \varphi$ is an order isomorphism exactly when the splitting map σ is positive.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
G_0 & \xrightarrow{n} & G_0 & \longrightarrow & G_0/n \oplus G_1[n] & \longrightarrow & G_1 & \xrightarrow{n} & G_1 \\
\| & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \| \\
G_0 & \xrightarrow{n} & G_0 & \xrightarrow{\rho} & G_n & \xrightarrow{\beta} & G_1 & \xrightarrow{n} & G_1,
\end{array}$$

notice that it is constructed to commute, and realise that it then follows from the Five Lemma that φ is bijective.

If $\mathrm{id} \oplus \varphi$ is an order isomorphism, it is especially positive. Hence, when $(x, y) \geq 0$ one gets as $(x, nG_0, y) \geq 0$ that $(x, \sigma(y)) = (x, \varphi(nG_0, y)) \geq 0$, i.e. $\mathrm{id} \oplus \sigma$ is positive.

On the other hand, assume that $\mathrm{id} \oplus \sigma$ is positive. If $(x_1, x_2 = nG_0, y) \geq 0$, then by definition $(x_1, x_2 + nG_0) \geq 0$ and $(x_1, y) \geq 0$, hence $(x_1, \rho(x_2)) \geq 0$ as $G_0 \oplus \operatorname{im} \rho$ has the quotient order and $(x_1, \sigma(y)) \geq 0$ as $\mathrm{id} \oplus \sigma$ is positive. As $G_0 \oplus G_n$ is a graded orded group, we may conclude that $(x_1, \varphi(x_2 + nG_0, y)) = (x_1, \rho(x_2) + \sigma(y)) \geq 0$, i.e. $\mathrm{id} \oplus \varphi$ is positive. If on the other hand $(x_1, \varphi(x_2 + nG_0, y)) \geq 0$, then $(x_1, y) = (x_1, \beta\varphi(x_2 + nG_0, y)) \geq 0$ as $G_0 \oplus \operatorname{im} \beta$ has the quotient order. As $\mathrm{id} \oplus \sigma$ is positive we get that $(x_1, \sigma(y)) \geq 0$, so as $(x_1, \rho(x_2) + \sigma(y)) = (x_1, \varphi(x_2 + nG_0, y)) \geq 0$ and as $G_0 \oplus G_n$ is a graded ordered group we get that also $(x_1, \rho(x_2)) \geq 0$, hence $(x_1, x_2 + nG_0) \geq 0$ as $G_0 \oplus \operatorname{im} \rho$ has the quotient order. I.e. $(x_1, x_2 + nG_0, y) \geq 0$ as desired, hence we may conclude that also $(\mathrm{id} \oplus \varphi)^{-1}$ is positive. ♡

7.3 Reduction results

It was shown in [Eil97] that the reduced invariant $(\mathrm{K}_*(-), \mathrm{K}_*(-)^+, \Sigma(-))$ is strongly complete for the class of real rank zero AD algebras with finitely many ideals. The strategy of proof is to construct a splitting of $\mathbb{K}(A;\infty)$ that respects the ideals of A in such a way that the splitting map is positive.

We try to copy this, but do so on the level of n-coefficient complexes, heavily relying on the range result 7.1, thus making some shortcuts.

Observation 7.12 Consider two AD algebras A and B with isomorphic ordered K_*-groups, and denote the isomorphism $(\varphi_0, \varphi_1)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$. Consider the two n-coefficient complexes $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$ and assume that for each of them one can find positive splitting maps. By considering the from the construction of 7.11 arrising diagram

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \xrightarrow{\rho} & \mathrm{K}_0(A;\mathbb{Z}/n) & \underset{\leftarrow\cdots}{\xrightarrow{\beta}} & \mathrm{K}_1(A) \\
\| & & \uparrow\cong & & \| \\
\mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A)/n \oplus \mathrm{K}_1(A)[n] & \longrightarrow & \mathrm{K}_1(A) \\
\downarrow\varphi_0 & & \downarrow\varphi_0\oplus\varphi_1 & & \downarrow\varphi_1 \\
\mathrm{K}_0(B) & \longrightarrow & \mathrm{K}_0(B)/n \oplus \mathrm{K}_1(B)[n] & \longrightarrow & \mathrm{K}_1(B) \\
\| & & \downarrow\cong & & \| \\
\mathrm{K}_0(B) & \xrightarrow{\rho} & \mathrm{K}(B;\mathbb{Z}/n) & \underset{\leftarrow\cdots}{\xrightarrow{\beta}} & \mathrm{K}_1(B)
\end{array}$$

and noticing that by construction the map $\mathrm{K}_0(A)/n \oplus \mathrm{K}_1(A)[n] \to \mathrm{K}_0(B)/n \oplus \mathrm{K}_1(B)[n]$ is an order-isomorphism, one sees that the isomorphism between $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ can be expanded to an isomorphism between $\mathbb{K}(A;n)$ and $\mathbb{K}(B;n)$, as a consequence of 7.11.

Ergo, the invariant $\mathbb{K}(A;n)$ carries more information about A than the invariant $(\mathrm{K}_*(A), \mathrm{K}*(A)^+, \Sigma(A))$ does exactly when the n-coefficient complex $\mathbb{K}(A;n)$ doesn't permit a positive splitting.

Observation 7.13 Consider an n-coefficient complex $\mathbb{G}$ and assume that G_1 is torsion-free. As $G_1[n] = 0$, $\beta = 0$ and the map $\sigma = 0$ is a positive splitting of $\mathbb{G}$, and $\mathrm{id} \oplus \rho$ is an order-isomorphism between $G \oplus G_0/n$ and $G_0 \oplus G_n$.

Now consider two real rank zero AD algebras A and B with $\mathrm{K}_1(A)$ and $\mathrm{K}_1(B)$ being torsion-free, and assume that a positive isomorphism $(\varphi_0, \varphi_1)\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ (with $\varphi_0(\Sigma(A)) = \Sigma(B)$) is given. As $(\varphi_0, \varphi_0 \oplus \varphi_1, \varphi_1)\colon \mathbb{K}(A;2) \to \mathbb{K}(B;2)$ is an isomorphism, we can by 6.8 find an isomorphism $\alpha\colon A \to B$ such that $\mathbb{K}(\alpha;2) = (\varphi_0, \varphi_0 \oplus \varphi_1, \varphi_1)$, in particular $\mathrm{K}_*(\alpha) = (\varphi_0, \varphi_1)$. I.e., we see that the invariant $(\mathrm{K}_*(-), \mathrm{K}_*(-)^+, \Sigma(-))$ is strongly complete for the class of real rank zero AD algebras with torsion-free $\mathrm{K}_1(-)$. Recall that the AT algebras have torsion-free $\mathrm{K}_1(-)$ as $\mathrm{K}(C(S^1)) = \mathbb{Z}$ is torsion-free; thus the AT algebras are classified by ordered K-theory.

To describe how the ideal structure of a real rank zero AD algebra A affects the order structure of $\mathrm{K}_0(A)$, we need the notion of order-ideals.

An *order-ideal* I in an ordered group (G, G^+) is a subgroup I of G satisfying that $I = I^+ - I^+$ (where $I^+ = I \cap G^+$) and having the hereditary property that if $g \in G$ satisfy $0 \leq g \leq i$ for some $i \in I$, then $g \in I$. Being an order-ideal is a transitive property, i.e. if I_1 is an order-ideal in I_2 and I_2 is an order-ideal in G, then I_1 is an order-ideal in G.

Example 7.14 Consider an ordered group (G, G^+) and $x \in G^+$. Then $I(x) = \{x' \in G \mid \exists n \in \mathbb{N} : -nx \leq x' \leq nx\}$ is the ordered ideal generated by x, i.e. the smallest order ideal containing x. One readily checks that $I(x)$ is an order ideal containing x. To see that $I(x)$ is the smallest such, we consider another order ideal I containing x. Given $x' \in I(x)$ we can write $-nx \leq x' \leq nx$ for some $n \in \mathbb{N}$, hence $0 \leq x' + nx \leq 2nx$, so as $2nx \in I$, $x' + nx \in I$, hence $x' \in I$ follows as $nx \in I$.

When the C^*-algebra A is σ-unital and of real rank zero and stable rank one, then, according to [Zha90, 2.3], there is an order-isomorphism between the lattice of ideals of A and the lattice of order-ideals in $\mathrm{K}_0(A)$, given by $I \mapsto \operatorname{im} \mathrm{K}_0(\iota)$ where $\iota \colon I \to A$ is the inclusion map.

Remark 7.15 The invariant considered in [ET] carries another order on the groups $\mathrm{K}_0(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ and $\mathrm{K}_*(A)$ than the ones defined in 4.4 and 4.1, namely the ones defined as follows: an element $(x, y) \in \mathrm{K}_*(A)$ is positive exactly when $x \geq 0$ and $y \in \operatorname{im} \mathrm{K}_1(\iota)$ where $\iota \colon I \to A$ is the inclusion of the unique ideal I of A that satisfy $\operatorname{im} \mathrm{K}_0(\iota) = I(x)$, and similarly $(x, z) \in \mathrm{K}_0(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ is positive when $z \in \operatorname{im} \mathrm{K}_0(\iota; \mathbb{Z} \oplus \mathbb{Z}/n)$.

According to [EE03], the ideal-basered ordering and the ordering defined in 4.4 and 4.1 coincide when A has real rank zero and stable rank one. And it was shown in [DE98, 5] that when considering real rank zero AD algebras A and B, then a morphism $(\varphi_0, \varphi, \varphi_1)$ from the complex $\mathbb{K}(A; n)$ to the complex $\mathbb{K}(B; N)$ is positive with respect to the ideal-based orders exactly when it is positive with respect to the orders defined in 4.4 and 4.1. Ergo, the invariant of [ET] is not the same as the one from the previous sections, but it is complete because the invariant of the previous sections is.

Proposition 7.16 Consider an n-coefficient complex $\mathbb{G}$. If G_0 has no nontrivial order-ideals, then there exists a positive splitting of $\mathbb{G}$.

Proof. According to 7.7 at least one splitting map σ exists. And we now show that any such splitting map σ will be positive: Let $(x, y) \in G_0 \oplus G_1[n]$ and assume that $(x, y) \geq 0$. If $x = 0$ then $y = 0$ as (x, y) then is a positive torsion-element, and $(x, \sigma(y)) \geq 0$ is then obvious. If $x > 0$ we have to put a tiny bit of efford into it. We know since $(x, y) \geq 0$ and since $G_0 \oplus \operatorname{im} \beta$ has the quotient order arrising from $G_0 \oplus G_n$, that we can find $z \in G_n$ such that $(x, z) \geq 0$ and $\beta(z) = y$. Then, $\sigma(y) - z \in \ker \beta = \operatorname{im} \rho$, hence we can find $w \in G_0$ such that $\rho(w) = \sigma(y) - z$. As G_0 has no nontrivial order-ideals, $G_0 \oplus G_0$ has the strict order arrising from G_0, hence $(x, w) \geq 0$ as $x > 0$, and therefore $(x, \rho(w)) \geq 0$. Therefore $(x, \sigma(y)) = (x, z + \rho(w)) \geq 0$ as $(x, z) \geq 0$ and $(x, \rho(w)) \geq 0$ and as $G_0 \oplus G_n$ is a graded ordered group. ♡

Lemma 7.17 Let $\mathbb{G}$ be an n-coefficient complex and let I_0 denote an order-ideal of G_0. Then the subcomplex

$$I_0 \xrightarrow{n} I_0 \xrightarrow{\rho|_{I_0}^{I_n}} I_n \xrightarrow{\beta|_{I_n}^{I_1}} I_1 \xrightarrow{n} I_1$$

where

$$I_n = \{z \in G_n \mid \exists x \in I_0 : (x, z) \geq 0\}, \quad I_1 = \{y \in G_1 \mid \exists x \in I_0 : (x, y) \geq 0\},$$

is an n-coefficient complex.

Proof. First of all we have to verify that the complex is well-defined, i.e. that I_n and I_1 are groups and that $\rho(I_0) \subseteq I_n$ and $\beta(I_n) \subseteq I_1$. As $G_0 \oplus G_n$ and $G_0 \oplus G_1$ are graded ordered groups, I_n and I_1 are groups, and one easily checks that $I_0 \oplus I_n$ and $I_0 \oplus I_1$ are ordered groups as I_0 is. Given $x \in I_0$, then clearly $(x, x) \geq 0$ as $x \in I(x)$, so as $\mathrm{id} \oplus \rho$ is positive, $(x, \rho(x)) \geq 0$, hence $\rho(x) \in I_n$ as $x \in I_0$. And given $z \in I_n$, we take an $x \in I_0$ such that $(x, z) \geq 0$, and as $\mathrm{id} \oplus \beta$ is positive, we see that $(x, \beta(z)) \geq 0$, hence $\beta(z) \in I_1$ as $x \in I_0$.

As for exactness in I_0, we see that $\ker(\rho|_{I_0}^{I_n}) = \ker \rho \cap I_0 = nG_0 \cap I_0 = nI_0$ where the last equality follows from the fact that I_0 is an order-ideal in G_0: given $g \in G_0$ satisfying $ng \in I_0$ we may write $ng = i_1 - i_2$ with $i_1, i_2 \in I_0^+$, so as $ng \leq i_1 \leq ni_1$ we see that $nh \geq 0$ where $h = i_1 - g$, ergo $h \geq 0$ as G_0 is unperforated, and $h \in I_0$ follows as I_0 is an order-ideal in G_0, ergo $g = i_1 - h \in I_0$.

Similarly, we obtain exactness in I_n by seeing that $\ker(\beta|_{I_n}^{I_1}) = \ker \beta \cap I_n = \mathrm{im}\, \rho \cap I_n = \rho(I_0)$ where the nontrivial part of the last equality is obtained as follows: given $x \in G_0$ such that $\rho(x) \in I_n$ we get by definition of I_n a $x' \in I_0$ such that $(x', \rho(x)) \geq 0$, and as $G_0 \oplus \mathrm{im}\, \rho$ has the quotient order arrising from $\mathrm{id} \oplus \rho$, we find a $x'' \in G_0$ such that $(x', x'') \geq 0$ and $\rho(x'') = \rho(x)$; as $(x', x'') \geq 0$, we see that $x'' \in I(x')$ where $I(x') \subseteq I_0$ as $x' \in I_0$, hence $\rho(x) = \rho(x'') \in \rho(I_0)$.

Finally, exactness in I_1 is obtained as $I_1[n] = G_1[n] \cap I_1 = \mathrm{im}\, \beta \cap I_1 = \beta(I_n)$ where the nontrivial part of the last equality is obtained as follows: given $z \in G_n$ such that $\beta(z) \in I_1$, we take a $x \in I_0$ such that $(x, \beta(z)) \geq 0$, and as $G_0 \oplus \mathrm{im}\, \beta$ has the quotient order from $\mathrm{id} \oplus \beta$ we get a $z' \in G_n$ such that $(x, z') \geq 0$ and $\beta(z') = \beta(z)$, and we see that $z' \in I_n$ as $(x, z') \geq 0$ and conclude that $\beta(z) = \beta(z') \in \beta(I_n)$.

Now, $I_0 \oplus I_n$ and $I_0 \oplus I_1$ are graded ordered groups as they are subgroups of such, and they restrict to the same order on I_0 as $G_0 \oplus G_n$ and $G_0 \oplus G_1$ restrict to the same order on G_0.

Notice that $I_0 \oplus I_1$ is an order-ideal in $G_0 \oplus G_1$: given $(x, y) \in G_0 \oplus G_1$ and $(x', y') \in I_0 \oplus I_1$ such that $0 \leq (x, y) \leq (x', y')$, we have in particular that $0 \leq x \leq x'$, hence $x \in I_0$ as I_0 is an order-ideal in G_0, and then $y \in I_1$ follows as $(x, y) \geq 0$, ergo $(x, y) \in I_0 \oplus I_1$. Using this, we see that $I_0 \oplus I_1$ has the Riesz interpolation property as $G_0 \oplus G_1$ has: given $x_i, y_j \in (I_0 \oplus I_1)^+$ that satisfy $x_1 + x_2 = y_1 + y_2$ we get $z_{ij} \in (G_0 \oplus G_1)^+$ such that $x_i = z_{i1} + z_{zi2}$ and $y_j = z_{1j} + z_{2j}$, and as $0 \leq z_{ij} \leq x_i$ and $x_i \in I_0 \oplus I_1$, $z_{ij} \in I_0 \oplus I_1$ holds.

Clearly, I_0 is unperforated as G_0 is. And, we see that $I_0 \oplus I_1$ is weakly unperforated as $G_0 \oplus G_1$ is: if $(x, y) \in I_0 \oplus I_1$ satisfy $m(x, y) \geq 0$, we use that $G_0 \oplus G_1$ is weakly unperforated to find a $t \in G_1[m]$ such that $(x, y + t) \geq 0$, so as $x \in I_0$ we get $y + t \in I_1$, hence $t \in I_1[m]$ holds.

Finally, we check that $I_0 \oplus \rho(I_0)$ and $I_0 \oplus \beta(I_n)$ has the quotient orders arrising from $\mathrm{id} \oplus \rho$ resp. $\mathrm{id} \oplus \beta$. As $\mathrm{id} \oplus \rho$ and $\mathrm{id} \oplus \beta$ are positive, we need only check that positive elements may be lifted to positive elements. Given $(x, z) \in I_0 \oplus \rho(I_0)$ with $(x, z) \geq 0$, we can find a $x' \in G_0$ such that $(x, x') \geq 0$ and $\rho(x') = z$, and as $(x, x') \geq 0$ we see that $x' \in I(x)$ where $I(x) \subseteq I_0$ as $x \in I_0$, hence $x' \in I_0$ as desired. Similarly, given $(x, y) \in I_0 \oplus \beta(I_n)$ with $(x, y) \geq 0$, we take a $z \in G_n$ with $(x, z) \geq 0$ and $\beta(z) = y$, and notice that $z \in I_n$ as $x \in I_0$. ♡

Lemma 7.18 Consider an n-coefficient complex $\mathbb{G}$ and an order-ideal I_0 in G_0, and define

$$I_n = \{z \in G_n \mid \exists x \in I_0 : (x, z) \geq 0\}$$

as in 7.17. Then there exist subgroups D' and D of G_0 satisfying

$$I_n = \rho(I_0) \oplus D', \quad G_n = \operatorname{im} \rho \oplus D, \quad D' \subseteq D.$$

Proof. By 7.7 we can find a subgroup $D' \subseteq I_n$ satisfying $\rho(I_0) \oplus D' = I_n$. As $\operatorname{im} \rho \cap I_n = \rho(I_0)$, $D' \cap \operatorname{im} \rho = 0$ and we can therefore find an $\operatorname{im} \rho$-high subgroup D of G_n that contains D'. As G_n, as well as its subgroups, is the direct sum of its p-components, it suffices to show that $G_n = \operatorname{im} \rho \oplus D$ holds when G_n is a p-group. When $pG_n = 0$ this follows from 7.6 as all $\mathbb{Z}/p$-modules are free and all short-exact sequences over $\mathbb{Z}/p$ therefore splits. The general case $p^m G_n = 0$ is then done by induction over m. ♡

Proposition 7.19 Consider an n-coefficient complex $\mathbb{G}$. If G_0 has exactly one nontrivial order-ideal, then there exists a positive splitting of $\mathbb{G}$.

Proof. Let I_0 denote the nontrivial order-ideal of G_0, and define

$$I_n = \{z \in G_n \mid \exists x \in I_0 : (x, z) \geq 0\}, \quad I_1 = \{y \in G_1 \mid \exists x \in I_0 : (x, y) \geq 0\}$$

and let $\mathbb{I}$ denote the resulting n-coefficient complex

$$I_0 \xrightarrow{\ n\ } I_0 \xrightarrow{\ \rho|_{I_0}^{I_n}\ } I_n \xrightarrow{\ \beta|_{I_n}^{I_1}\ } I_1 \xrightarrow{\ n\ } I_1$$

as in 7.17. By 7.18 there exist splitting maps σ' and σ that makes also the dotted squares in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \rho(I_0) & \longrightarrow & I_n & \underset{\sigma'}{\rightleftarrows} & I_1[n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{im} \rho & \longrightarrow & G_n & \underset{\sigma}{\rightleftarrows} & G_1[n] & \longrightarrow & 0
\end{array}$$

commute, the vertical maps being the inclusions. As a nontrivial order-ideal in I_0 would be a nontrivial order-ideal in G_0 other than I_0, I_0 has no nontrivial order-ideals, hence 7.16 assures us that the splitting map σ' is positive.

We can now show that the splitting map σ is positive. Let $(x, y) \in G_0 \oplus G_1[n]$ and assume that $(x, y) \geq 0$. If $x \in I_0$, then $y \in I_1$ per definition of I_1, hence $(x, \sigma(y)) = (x, \sigma'(y)) \geq 0$.

If $x \notin I_0$ then $I(x) = G_0$, so $(x, x') \geq 0$ holds for any $x' \in G_0$. As $(x, y) \geq 0$, we can find a $z \in G_n$ satisfying $(x, z) \geq 0$ and $\beta(z) = y$. As $\sigma(y) - z \in \ker\beta = \operatorname{im}\rho$, we can find $x' \in G_0$ satisfying $\sigma(y) = z + \rho(x')$. As $(x, x') \geq 0$, $(x, \rho(x')) \geq 0$, so as $(x, z) \geq 0$ and $G_0 \oplus G_n$ is a graded ordered group, $(x, \sigma(y)) \geq 0$ follows. ♡

Corollary 7.20 The invariant $(\mathrm{K}_*(-), \mathrm{K}_*(-)^+, \Sigma(-))$ consisting of ordered $\mathrm{K}_*(-)$ together with the scale $\Sigma(-)$ is a strongly complete invariant for the class of real rank zero AD algebras that have bounded torsion in $\mathrm{K}_1(-)$ and have atmost one nontrivial ideal.

Proof. As all AD algebras have stable rank one, a real rank zero AD algebra A has atmost one nontrivial ideal exactly when $\mathrm{K}_0(A)$ has atmost one nontrivial order-ideal. Given real rank zero AD algebras A and B with bounded torsion in $\mathrm{K}_1(A)$ and $\mathrm{K}_1(B)$ and a scale- and order-isomorphism (φ_0, φ_1) from $\mathrm{K}_*(A)$ to $\mathrm{K}_*(B)$, the existence of an isomorphism $\alpha\colon A \to B$ satisfying $\mathrm{K}_*(\alpha) = (\varphi_0, \varphi_1)$ follows from 7.16 and 7.19 by considering an n suffieciently large to kill the torsion in $\mathrm{K}_1(A)$ and $\mathrm{K}_1(B)$ and then combining 6.8 and 7.12, as in 7.13. ♡

Remark 7.21 The article [ET] also contains a range result for the invariant $\mathbb{K}(-;\infty)$. It seems possible, by the exact same methods as above, to use this range result instead, together with the strong completeness of $\mathbb{K}(-;\infty)$, to get rid of the condition on $\operatorname{tor}\mathrm{K}_1(-)$ in 7.20.

7.4 Irreducibility of $\mathbb{K}(-;n)$

In [Eil95] and [DE99], examples are constructed to show that the invariant $\mathbb{K}(-;n)$ cannot be reduced.

For instance, to show necessity of the Bockstein map β_n in [DE99], nonisomorphic real rank zero AD algebras A and B are constructed such that $n\operatorname{tor}\mathrm{K}_1(A) = 0$ and such that there exist isomorphisms $\varphi_0\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$, $\varphi\colon \mathrm{K}_0(A;\mathbb{Z}/n) \to \mathrm{K}_0(B;\mathbb{Z}/n)$ and $\varphi_1\colon \mathrm{K}_1(A) \to \mathrm{K}_1(B)$ satisfying that the diagram

$$\begin{array}{ccc} \mathrm{K}_0(A) & \xrightarrow{\rho_n^A} & \mathrm{K}_0(A;\mathbb{Z}/n) \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_0} \\ \mathrm{K}_0(B) & \xrightarrow{\rho_n^B} & K_0(B;\mathbb{Z}/n) \end{array}$$

commutes and that $\varphi_0 \oplus \varphi$ and $\varphi_0 \oplus \varphi_1$ are order-isomorphisms.

It must be possible to show irreducibility of the invariant via the range result 7.1. To show necessity of the Bockstein map β_n, for instance, it will suffice to construct an n-coefficient complex $\mathbb{G}$ that omits a map $\beta'\colon G_n \to G_1$ that makes

$$G_0 \xrightarrow{n} G_0 \xrightarrow{\rho} G_n \xrightarrow{\beta'} G_1 \xrightarrow{n} G_1$$

an n-coefficient complex $\mathbb{G}'$ and doesn't admit an isomorphism $(\varphi_0, \varphi, \varphi_1)$ between $\mathbb{G}$ and $\mathbb{G}'$. Because then by 7.1, real rank zero AD algebras A and B with $\mathbb{K}(A; n) = \mathbb{G}$ and $\mathbb{K}(B; n) = \mathbb{G}'$ exist.

References

[Bla98] Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR MR1656031 (99g:46104)

[Bla06] ———, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR MR2188261 (2006k:46082)

[Böd79] Carl-Friedrich Bödigheimer, *Splitting the Künneth sequence in K-theory*, Math. Ann. **242** (1979), no. 2, 159–171. MR MR537958 (80k:55013)

[Böd80] C.-F. Bödigheimer, *Splitting the Künneth sequence in K-theory. II*, Math. Ann. **251** (1980), no. 3, 249–252. MR MR589253 (81k:55007)

[Con90] John B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR MR1070713 (91e:46001)

[DE98] Marius Dădărlat and Søren Eilers, *Compressing coefficients while preserving ideals in K-theory for C^*-algebras*, K-Theory **14** (1998), no. 3, 281–304. MR MR1633513 (99h:46132)

[DE99] ———, *The Bockstein map is necessary*, Canad. Math. Bull. **42** (1999), no. 3, 274–284. MR MR1703687 (2000d:46070)

[DG97] M. Dădărlat and G. Gong, *A classification result for approximately homogeneous C^*-algebras of real rank zero*, Geom. Funct. Anal. **7** (1997), no. 4, 646–711. MR MR1465599 (98j:46062)

[DL94] M. Dădărlat and A. Loring, *Extensions of certain real rank zero C^*-algebras*, Ann. Inst. Fourier (Grenoble) **44** (1994), no. 3, 907–925. MR MR1303887 (96b:46075)

[EE93] George A. Elliott and David E. Evans, *The structure of the irrational rotation C^*-algebra*, Ann. of Math. (2) **138** (1993), no. 3, 477–501. MR MR1247990 (94j:46066)

[EE03] Søren Eilers and George A. Elliott, *The Riesz property for the K_*-group of a C^*-algebra of minimal stable and real rank*, C. R. Math. Acad. Sci. Soc. R. Can. **25** (2003), no. 4, 108–113. MR MR2013159 (2004g:46084)

[Eil95] Søren Eilers, *Invariants for AD algebras*, Ph.D. thesis, University of Copenhagen, December 1995.

[Eil97] ———, *Künneth splittings and classification of C^*-algebras with finitely many ideals*, Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 81–90. MR MR1424956 (97j:46057)

[Ell76] George A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44. MR MR0397420 (53 #1279)

[Ell93] ———, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219. MR MR1241132 (94i:46074)

[ET] Søren Eilers and Andrew S. Toms, *Inductive limits of K-theoretic complexes with torsion coefficients*, To appear in Journal of K-theory.

[Fox04] Hans-Bjørn Foxby, *Homological algebra*, December 20 2004, course notes.

[Fuc70] László Fuchs, *Infinite abelian groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970. MR MR0255673 (41 #333)

[Goo86] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs, vol. 20, American Mathematical Society, Providence, RI, 1986. MR MR845783 (88f:06013)

[Mur90] Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc. Boston, MA, 1990. MR MR1074574 (91m:46084)

[Rie83] Marc A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46** (1983), no. 2, 301–333. MR MR693043 (84g:46085)

[RLL00] M. Rørdam, F. Larsen, and N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000. MR MR1783408 (2001g:46001)

[RS87] Jonathan Rosenberg and Claude Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), no. 2, 431–474. MR MR894590 (88i:46091)

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR MR1269324 (95f:18001)

[Zha90] Shuang Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras*, J. Operator Theory **24** (1990), no. 2, 209–225. MR MR1150618 (93b:46116)

Addendum

There was a slight error in the proof of lemma 7.18. A corrected proof here follows.

Observation 7.22 For any n, $\mathbb{Z}/n$ is an injective $\mathbb{Z}/n$-module. To see this, we use Baer's criterion, so let $\mathfrak{a}$ be an ideal in $\mathbb{Z}/n$ and let $\varphi\colon \mathfrak{a} \to \mathbb{Z}/n$ be some homomorphism. By Noether's isomorphisms $\mathfrak{a} = m(\mathbb{Z}/n)$ for some m dividing n. As $\frac{n}{m}\varphi(m) = 0$, it follows that m divides $\varphi(m)$ and we may therefore expand φ to a map $\mathbb{Z}/n \to \mathbb{Z}/n$ by $1 \mapsto \frac{\varphi(m)}{m}$. As the ring $\mathbb{Z}/n$ is Noetherian, we have now shown that $\mathbb{Z}/n$ is quasi-Frobenius.

The following proof is inspired by [Fox04, 18.23 & 20.16 & 20.18].

Lemma 7.23 Given a torsion-free group G, G/n is an injective $\mathbb{Z}/n$-module.

Proof. We may identify G/n with $\mathbb{Z}/n \otimes G$. As we use Baer's criterion, we let an ideal $\mathfrak{a}$ in $\mathbb{Z}/n$ be given and let $\iota\colon \mathfrak{a} \to \mathbb{Z}/n$ denote its embedding into $\mathbb{Z}/n$, and desire to proof surjectivity of $\operatorname{Hom}(\iota, \mathbb{Z}/n \otimes G)$.

Define functors S and T by $S = \operatorname{Hom}(-, \mathbb{Z}/n \otimes G)$ and $T = \operatorname{Hom}(-, \mathbb{Z}/n) \otimes G$, and notice that we have a natural transformation $\Theta\colon T \to S$ defined on a group A by

$$\begin{aligned} \Theta^A\colon T(A) &\to S(A) \\ \varphi \otimes g &\mapsto (a \mapsto \varphi(a) \otimes g) \end{aligned}$$

and satisfying that $\Theta^{\mathbb{Z}/n}$ is an isomomorphism.

By Noether's isomorphisms $\mathfrak{a} = m(\mathbb{Z}/n)$ for some m dividing n, so we may consider the following presentation of $\mathfrak{a}$

$$\mathbb{Z}/n \xrightarrow{\frac{n}{m}} \mathbb{Z}/n \xrightarrow{m} \mathfrak{a} \longrightarrow 0$$

and by applying the functors S and T we obtain the following commuting diagram

$$\begin{array}{ccccccc} S(\mathbb{Z}/n) & \longleftarrow & S(\mathbb{Z}/n) & \longleftarrow & S(\mathfrak{a}) & \longleftarrow & 0 \\ \cong\big\uparrow \Theta^{\mathbb{Z}/n} & & \cong\big\uparrow \Theta^{\mathbb{Z}/n} & & \big\uparrow \Theta^{\mathfrak{a}} & & \\ T(\mathbb{Z}/n) & \longleftarrow & T(\mathbb{Z}/n) & \longleftarrow & T(\mathfrak{a}) & \longleftarrow & 0 \end{array}$$

where the top row is exact as the functor S is left-exact, while the bottom row is exact as T is an exact functor of $\mathbb{Z}/n$-modules since $\operatorname{Hom}(-, \mathbb{Z}/n)$ by 7.22 is an exact functor of $\mathbb{Z}/n$-modules and $- \otimes G$ is an exact functor of $\mathbb{Z}$-modules. By applying Five Lemma we obtain that $\Theta^{\mathfrak{a}}$ is an isomorphism.

As T is an exact functor of $\mathbb{Z}/n$-modules, $T(\iota)$ is surjective, hence $S(\iota) = \Theta^{\mathfrak{a}} T(\iota) (\Theta^{\mathfrak{a}})^{-1}$ is also surjective, as desired. ♡

Corollary 7.24 For any n-coefficient complex $\mathbb{G}$, $\operatorname{im} \rho$ is an injective $\mathbb{Z}/n$-module.

Remark 7.25 Any $\mathbb{Z}$-homomorphism between $\mathbb{Z}/n$-modules is a $\mathbb{Z}/n$-homomorphism, and any $\mathbb{Z}/n$-homomorphism is a $\mathbb{Z}$-homomorphism. As G_n is a $\mathbb{Z}/n$-module for any n-coefficient complex $\mathbb{G}$, if follows immediately from 7.24 that n-coefficient complexes split. Hence, the hard work that was done to establish 7.7 has been a waste of time.

Lemma 7.18 Consider an n-coefficient complex $\mathbb{G}$ and an order-ideal I_0 in G_0, and define

$$I_n = \{z \in G_n \mid \exists x \in I_0 : (x, z) \geq 0\}$$

as in 7.17. Then there exist subgroups D' and D of G_0 satisfying

$$I_n = \rho(I_0) \oplus D', \quad G_n = \operatorname{im} \rho \oplus D, \quad D' \subseteq D.$$

Proof. By 7.17 and 7.7 we can find a subgroup $D' \subseteq I_n$ satisfying $\rho(I_0) \oplus D' = I_n$. As $\operatorname{im} \rho \cap I_n = \rho(I_0)$, $D' \cap \operatorname{im} \rho = 0$ and we can therefore find an $\operatorname{im} \rho$-high subgroup D of G_n that contains D'. As $\operatorname{im} \rho$ by 7.24 is an injective $\mathbb{Z}/n$-module, any short-exact sequence of $\mathbb{Z}/n$-modules with $\operatorname{im} \rho$ appearing as the left-most module will split. Hence it follows from 7.6 that $\operatorname{im} \rho \oplus D = G_n$. ♡