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Classification of nonsimple C^*-algebras of real rank zero

PhD thesis by

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Abstract

This thesis deals with classification of nonsimple C^*-algebras of real rank zero, and whether filtered K-theory is a suitable invariant for this purpose.

As a consequence of the result of E. Kirchberg for purely infinite, nuclear C^*-algebras with a finite primitive ideal space, it suffices to lift isomorphisms on filtered K-theory to ideal related KK-equivalences to achieve the desired classification result. Results by R. Meyer and R. Nest, and by R. Bentmann and M. Köhler, describe for exactly which finite primitive ideal spaces this is possible for general C^*-algebras.

The main question throughout the thesis is the following: is it possible to achieve the desired classification result for arbitrary finite primitive ideal spaces by restricting to C^*-algebras of real rank zero that possibly satisfy further restrictions on K-theory? The thesis consists of an account of the relevant theory and the relevant results, plus two articles.

The smallest primitive ideal spaces that do not admit classification of general C^*-algebras, are six four-point spaces. In the first article (with G. Restorff and E. Ruiz), these six four-point spaces are examined, and it is shown that for four of these spaces, isomorphisms are liftable for C^*-algebras of real rank zero.

In the second article (with R. Bentmann and T. Katsura) it is shown that for real rank zero C^*-algebras whose subquotients have free K_1-groups, isomorphisms are liftable also for a fifth of the spaces. In this article, the range of filtered K-theory is determined for real rank zero graph algebras over primitive ideal spaces that admit classification. As a consequence of completeness of filtered K-theory combined with this range result, one can conclude that real rank zero extensions of stabilized Cuntz-Krieger algebras are stabilized Cuntz-Krieger algebras, provided the primitive ideal space permits classification.

The following is a Danish translation of the abstract as required by the rules of the University of Copenhagen.

Resumé

Denne afhandling omhandler klassifikation af ikke-simple C^*-algebraer af reel rang nul og hvorvidt invarianten filtreret K-teori er passende til formålet.

Som en konsekvens af E. Kirchbergs resultat for rent uendelige, nukleære C^*-algebraer med endeligt primitivt idealrum, er det tilstrækkeligt at kunne løfte isomorfier på den filtrerede K-teori til idealrelateret KK-ækvivalenser for at opnå det ønskede klassifikationsresultat. Resultater af R. Meyer og R. Nest samt R. Bentmann og M. Köhler beskriver for præcis hvilke endelige primitive idealrum dette er muligt for generelle C^*-algebraer.

Det gennemgående spørgsmål i afhandlingen er følgende: er det muligt at opnå det ønskede klassifikationsresultat for vilkårlige endelige primitive idealrum såfremt man restringerer til C^*-algebraer af reel rang nul og eventuelt med yderligere K-teoretiske restriktioner? Afhandlingen består af en redegørelse for den relevante teori og de relevante resultater samt to artikler.

De mindste primitive idealrum der ikke tillader klassifikation af generelle C^*-algebraer, er seks firepunktsrum. I den første artikel (med G. Restorff og E. Ruiz) undersøges disse seks firepunktsrum, og det vises at for de fire kan isomorfier løftes såfremt C^*-algebraerne har reel rang nul.

I den anden artikel (med R. Bentmann og T. Katsura) vises det at såfremt der restringeres til C^*-algebraer af reel rang nul hvis subkvotienter alle har frie K_1-grupper, kan isomorfier løftes for yderligere et rum. I artiklen bestemmes desuden, for de primitive idealrum der tillader klassifikation, billedet af filtreret K-teori for grafalgebraer af reel rang nul. Som en konsekvens af fuldstændighed af filtreret K-teori kombineret med dette billedresultat kan det sluttes at ekstensioner af reel rang nul af stabiliserede Cuntz-Krieger-algebraer er stabiliserede Cuntz-Krieger-algebraer, givet det primitive idealrum tillader klassifikation.

Preface

This text constitutes my thesis for the PhD degree in mathematics from the PhD School of Science at the Faculty of Science, University of Copenhagen where I have been enrolled from May 2008 to January 2012. I started out studying stably finite C^*-algebras and their automorphism groups, but September 2009 I switched focus to nonsimple, purely infinite C^*-algebras, and this thesis focuses solely on the latter.

The thesis consists of the two articles *Filtrated K-theory of real rank zero C^*-algebras*, [**ARR**], with Gunnar Restorff, and Efren Ruiz, and *Reduction of filtered K-theory and a characterization of Cuntz-Krieger algebras*, [**ABK**], with Rasmus Bentmann, and Takeshi Katsura, together with an account, Chapters 2 and 3, of the theory and results that the articles are based on and are a continuation of. The first article, [**ARR**], has been submitted to *International Journal of Mathematics*, while the second article, [**ABK**], is still a preprint.

The main results of the two articles are quoted in Chapters 2 and 3, and it is possible to read Chapters 1 to 3 without reading the articles. Please note that some of the quoted results, both those from the two articles and those by others, are quoted in a weaker form to ease notation and improve readability.

The subject of the articles is filtered K-theory of real rank zero C^*-algebras and of graph algebras. Chapter 2 is therefore on filtered K-theory and filtered K-theory of real rank zero C^*-algebras, while Chapter 3 is on graph algebras and filtered K-theory of graph algebras.

Chronological course

When Ralf Meyer and Ryszard Nest introduced their counterexample over the space $\mathcal{W}$ in the fall of 2008, it killed almost all hope in filtered K-theory as a classifying functor. My advisor, Søren Eilers, raised the question of whether their counterexample had real rank zero. The answer was that the counterexample itself did not have real rank zero but that there existed a suitably nice real rank zero C^*-algebra over $\mathcal{W}$ whose filtered K-theory had projective dimension 2, and it was believed that this made it most likely that real rank zero counterexamples existed.

In an attempt to understand what properties of Cuntz-Krieger algebras made their classification possible, Gunnar Restorff, Efren Ruiz, and I examined the filtered K-theory of a real rank zero C^*-algebra over $\mathcal{W}$ and as a result proved in the fall of 2010 that filtered K-theory does classify the real rank zero C^*-algebras over $\mathcal{W}$ that are tight, stable, purely infinite, nuclear, separable and have all simple subquotients in the bootstrap class.

Continuing with the space $\mathcal{Y}$ over which Rasmus Bentmann had constructed a counterexample using the methods of Ralf Meyer and Ryszard Nest, we got the same positive result. For the space $\mathcal{D}$ over which Rasmus Bentmann also had constructed a counterexample, our methods did not apply, and eventually I calculated the filtered K-theory of the constructed counterexample and discovered disappointingly that the counterexample had real rank zero.

Since one can construct plenty of Cuntz-Krieger algebras with $\mathcal{D}$ as their primitive ideal space, it was natural to take another property of the Cuntz-Krieger algebras into account. For the space $\mathcal{D}$, the position of the K_1-groups in the filtered K-theory of a real rank zero C^*-algebra made it likely that freeness of these groups was sufficient or at least important, and Takeshi Katsura noticed in the spring of 2011 that the methods he, Rasmus Bentmann, and I were using to determine the range of filtered K-theory for graph algebras, applied to prove classification of C^*-algebras over $\mathcal{D}$ with the K-theory of a graph algebra using the reduced filtered K-theory.

By introducing the notion of unique path property for a finite primitive ideal space, we are beginning to be able to describe what causes the existence of real rank zero counterexamples.

Acknowledgements

Most of my work was only made possible through the knowledge I gained during my stay at Georg-August-Universität Göttingen in the spring and early summer of 2010, and I am most grateful to Ralf Meyer and the Mathematisches Institut at Georg-August-Universität Göttingen for their kind hospitality.

Thanks are due to the NordForsk Research Network "Operator Algebras and Dynamics" (grant #11580) as it supported my short and fruitful visits to the University of the Faroe Islands in September 2009 and September 2011 to work with Gunnar Restorff. Thanks are also due to the Faculty of Science at the University of the Faroe Islands, and to Gunnar Restorff and his wonderful family for kind hospitality.

The research environment at the Department of Mathematical Sciences at University of Copenhagen has flourished after the inauguration of the Centre for Symmetry and Deformation in January 2010, and therefore I am also grateful to the Centre for Symmetry and Deformation and its funder the Danish National Research Foundation (DNRF).

I have really enjoyed having coauthors, and I would like to thank Rasmus Bentmann, Takeshi Katsura, Gunnar Restorff, and Efren Ruiz for the many discussions we had — mathematical and nonmathematical. I would also like to thank Ryszard Nest, Mikael Rørdam, Adam Sørensen, and Hannes Thiel for enlightening conversations.

Finally, I am very grateful to my advisor Søren Eilers for being most supportive and inspiring, and for catching my fall.

Sara Arklint
Copenhagen, January 2012

Contents

CHAPTER 1

Introduction

1.1. On filtered K-theory

The strong classification of Kirchberg algebras, i.e., simple, stable, purely infinite, nuclear, separable C^*-algebras, in the bootstrap class consists of two parts, namely the result due to E. Kirchberg and N. C. Phillips saying that any KK-equivalence between Kirchberg algebras lifts to a $*$-isomorphism, and the Universal Coefficient Theorem of J. Rosenberg and C. Schochet by which one can lift any isomorphism on the K-theory of separable C^*-algebras in the bootstrap class to a KK-equivalence. Shortly after proving the classification result for Kirchberg algebras, E. Kirchberg generalized the result to X-equivariant KK-theory by proving that for tight, stable, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebras over a space X, any X-equivariant KK-equivalence lifts to a X-equivariant $*$-isomorphism.

The classification of nonsimple, stable, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebras would therefore be complete if one could establish an X-equivariant Universal Coefficient Theorem. This was done by A. Bonkat for separable C^*-algebras A with exactly one nontrivial ideal I and the invariant consisting of the six-term sequence in K-theory induced by the extention $I \hookrightarrow A \twoheadrightarrow A/I$. The result of A. Bonkat thereby gave a strong version of the classification of stable, purely infinite, nuclear, separable C^*-algebras with exactly one nontrivial ideal which was due to M. Rørdam who had introduced the invariant.

Inspired by the result of M. Rørdam, G. Restorff classified a certain class of nonsimple, unital, purely infinite, nuclear, separable C^*-algebras with finitely many ideals, namely the Cuntz-Krieger algebras satisfying property (II), using a generalized version of the invariant of M. Rørdam that consisted of six-term sequences in K-theory induced by extensions of ideals in the C^*-algebra. For separable C^*-algebras with exactly two nontrivial ideals, G. Restorff established a Universal Coefficient Theorem for his invariant.

Shortly after, R. Meyer and R. Nest introduced filtered K-theory for C^*-algebras with finitely many ideals, and established a Universal Coefficient Theorem for their invariant under some restrictions on the primitive ideal space of the C^*-algebras. This filtered K-theory, which R. Meyer and R. Nest denotes filtrated K-theory, includes the six-term sequences in K-theory induced by all extensions of subquotients of the C^*-algebra and thereby generalizes the invariants mentioned above. Most disappointing, R. Meyer and R. Nest also constructed two nonisomorphic stable, purely infinite, nuclear, separable C^*-algebras with the same finite primitive ideal space and isomorphic filtered K-theory, showing that the intuitively right invariant is not sufficient. Later, R. Bentmann and M. Köhler used the methods of R. Meyer and R. Nest to establish exactly which finite primitive ideal spaces admit a Universal Coefficient Theorem and classification of stable, purely infinite, nuclear, separable C^*-algebras.

As any finite T_0-space can be realized as the primitive ideal space of a Cuntz-Krieger algebra, the question is naturally: why are Cuntz-Krieger algebras classified by filtered K-theory when general purely infinite, nuclear, separable C^*-algebras with finitely many ideals are not?

In an attempt to answer this question, filtered K-theory of C^*-algebras of real rank zero, and of graph algebras, is studied in this thesis.

1.2. On real rank zero

Intuitively, real rank zero guarantees that the C^*-algebra has many projections and thereby that its K-theory carries a lot of information. All Kirchberg algebras have real rank zero, but not all nonsimple, purely infinite, nuclear, separable C^*-algebras do.

It is at the same time surprising and not suprising that real rank zero turns out to play a role in the classification of nonsimple, purely infinite C^*-algebras.

Not surprising, since real rank zero played a significant role in the classification of nonsimple, stably finite C^*-algebras. In classification of stably finite C^*-algebras, one considers either the simple case or the nonsimple real rank zero case, e.g., in the classification of simple AT algebras by G. A. Elliott using the Elliott invariant, or in the classification of real rank zero AH algebras of slow dimension growth by M. Dadarlat and G. Gong using ordered total K-theory.

But also surprising, since in the classification of nonsimple, stably finite C^*-algebras, the role of real rank zero is to guarantee that the ordered K_0-group contains enough information to keep track of the ideal structure of the C^*-algebra. For stably finite C^*-algebras of real rank zero, there is an isomorphism between the lattice of ideals in the C^*-algebra and the lattice of order ideals in the ordered K_0-group. But for purely infinite C^*-algebras, all elements in the K_0-group are positive, and the invariant introduced to hopefully classify nonsimple, purely infinite C^*-algebras is filtered K-theory which already keeps track of the ideal structure.

A nonsimple, purely infinite, separable, nuclear C^*-algebra with finitely many ideals has real rank zero if and only if its filtered K-theory satisfies the following condition: all boundary maps from even to odd K-groups vanish. This follows from the fact that all simple subquotients of such a C^*-algebra are Kirchberg algebras and therefore have rel rank zero, combined with the following result of L. G. Brown and G. K. Pedersen:

THEOREM 1.2.1 ([**BP91**, 3.14]). *Let $I \hookrightarrow A \twoheadrightarrow A/I$ be an extension of C^*-algebras. Then A has real rank zero if and only if I and A/I have real rank zero and projections in A/I lift to projections in A.*

1.3. On graph algebras

Graph algebras is a relevant class of C^*-algebras to study for many reasons. On one hand it is a large class containing, e.g., both the AF algebras and the Cuntz-Krieger algebras, and on the other hand it is a well-behaved and well-controlled class.

Several properties on the C^*-algebraic level — e.g., pure infiniteness, ideal structure, real rank zero — correspond to properties of the graph. If one desires to construct a C^*-algebra with certain properties, one can therefore do it by constructing a graph with the corresponding properties. As an example, in [**EK**], S. Eilers and T. Katsura provide a counterexample to a conjecture concerning semiprojectivity by translating a relevant property of C^*-algebras to a property of graphs.

Most classification results deal with either stably finite or purely infinite C^*-algebras. Within the class of stably finite C^*-algebras there is, e.g., the classification of AF algebras by O. Bratteli and G. A. Elliot, or the result of M. Dadarlat and G. Gong mentioned earlier. Within the class of purely infinite C^*-algebras, there is, e.g., the results of E. Kirchberg and N. C. Phillips, or M. Rørdam, mentioned earlier. The graph algebras, however, are a mix of purely infinite and stably finite C^*-algebras, in the way that a simple subquotient of a graph algebra is either a Kirchberg algebra, hence purely infinite, or an AF algebra, hence stably finite. This makes the graph algebras a suitable test class for an invariant constructed to handle both purely infinite and stably finite C^*-algebras. S. Eilers, G. Restorff, and E. Ruiz have conjectured that graph algebras with finitely many ideals are classified by ordered filtered K-theory.

When only dealing with the purely infinite case, the purely infinite graph algebras provide a fairly large test class for filtered K-theory — the only restriction on the filtered K-theory apparently being freeness of all K_1-groups and the vanishing of maps caused by real rank zero. Furthermore, when it comes to filtered K-theory, a very useful property of graph algebras is that there is a straightforward and easy algorithm for computing their ordered filtered K-theory. Usually, larger K-theoretic invariants are not that easy to calculate.

1.4. Main contributions

By a result of R. Bentmann and M. Köhler, the filtered K-theory over a finite T_0-space X admits a Universal Coefficient Theorem if and only if X is a so-called accordion space. All spaces with three or less points are accordion spaces, and there are up to homeomorphism six nonaccordion four-point spaces; in the following chapters they will be denoted $\mathcal{W}$, $\mathcal{W}^{\mathrm{op}}$, $\mathcal{Y}$, $\mathcal{Y}^{\mathrm{op}}$, $\mathcal{D}$, and $\mathcal{S}$.

R. Meyer and R. Nest, and R. Bentmann have for all $X \in \{\mathcal{W}, \mathcal{Y}, \mathcal{D}, \mathcal{S}\}$ constructed counterexamples to classification of purely infinite C^*-algebras over X, i.e., constructed non-$KK(X)$-equivalent, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class, and with isomorphic filtered K-theory.

In [**ABK**], the notion of X having the unique path property, a generalization of accordion spaces, is introduced, and for such X a reduction $\mathrm{FK}_{\mathcal{B}}$ of filtered K-theory is defined. The spaces $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{S}$ have the unique path property while $\mathcal{D}$ does not. In [**ABK**], the reduction $\mathrm{FK}_{\mathcal{R}}$ of filtered K-theory that was introduced by G. Restorff to classify Cuntz-Krieger algebras, is also studied.

The main points and contributions in the thesis are the following.

- The constructed counterexamples over $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{S}$ are not of real rank zero. The constructed counterexample over $\mathcal{D}$ is of real rank zero.
- Let $X \in \{\mathcal{W}, \mathcal{W}^{\mathrm{op}}, \mathcal{Y}, \mathcal{Y}^{\mathrm{op}}\}$, then for *real rank zero*, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class, any isomorphism on filtered K-theory FK lifts to an X-equivariant $*$-isomorphism.
- For *real rank zero*, tight, stable, purely infinite, nuclear, separable C^*-algebras over $\mathcal{D}$ with all simple subquotients in the bootstrap class and with *free K_1-groups*, any isomorphism on $\mathrm{FK}_{\mathcal{R}}$ lifts to an $\mathcal{D}$-equivariant $*$-isomorphism.
- Assume X has the boundary decomposition property, then for *real rank zero* C^*-algebras over X, any isomorphism on $\mathrm{FK}_{\mathcal{B}}$ extends uniquely to an isomorphism on $\mathrm{FK}_{\mathcal{ST}}$. For X an accordion space or one of the spaces $\mathcal{W}$, $\mathcal{W}^{\mathrm{op}}$, $\mathcal{Y}$, and $\mathcal{Y}^{\mathrm{op}}$, this means that for real rank zero, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class, any isomorphism on $\mathrm{FK}_{\mathcal{B}}$ lifts to an X-equivariant $*$-isomorphism.
- Assume X has the boundary decomposition property, then for *real rank zero* C^*-algebras with *free K_1-groups* for all simple subquotients, any isomorphism on $\mathrm{FK}_{\mathcal{R}}$ extends (nonuniquely) to an isomorphism on $\mathrm{FK}_{\mathcal{ST}}$. For X an accordion space or one of the spaces $\mathcal{W}$, $\mathcal{W}^{\mathrm{op}}$, $\mathcal{Y}$, and $\mathcal{Y}^{\mathrm{op}}$, this means that for real rank zero, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class and with free K_1-groups, any isomorphism on $\mathrm{FK}_{\mathcal{R}}$ lifts to an X-equivariant $*$-isomorphism.
- For a C^*-algebra A over any finite T_0-space X, $\mathrm{FK}_{\mathcal{R}}(A)$ is isomorphic to $\mathrm{FK}_{\mathcal{R}}(B)$ for B a tight, purely infinite graph algebra if and only if $K_1(A(x))$ is free for all $x \in X$. Combined with the above result, this determines the range of $\mathrm{FK}_{\mathcal{ST}}$ for real rank zero graph algebras over any finite space X with the boundary decomposition property.

The notion of boundary decomposition property is a technical condition on spaces with the unique path property and will be introduced in Chapter 3. The invariant $\mathrm{FK}_{\mathcal{ST}}$ is referred to as concrete filtered K-theory and will be introduced in Chapter 2. For accordion spaces and the six nonaccordion four-points spaces it is known that $\mathrm{FK}_{\mathcal{ST}}$ equals FK, but it is unknown whether there exists a finite T_0-space X for which $\mathrm{FK}_{\mathcal{ST}}$ is strictly coarser than FK.

1.5. Unanswered questions

The following questions are still unanswered but it seems likely that they have a positive answer. Let X be a finite T_0-space.

- Do all isomorphisms on $\mathrm{FK}_{\mathcal{R}}$ extend to isomorphisms on FK for real rank zero C^*-algebras A over X that have the property that $K_1(A(x))$ is free for all $x \in X$?
- Assume X has the unique path property. Do all isomorphisms on $\mathrm{FK}_{\mathcal{B}}$ lift to X-equivariant $*$-isomorphisms for real rank zero, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class?
- Do all isomorphisms on $\mathrm{FK}_{\mathcal{R}}$ lift to X-equivariant $*$-isomorphisms for real rank zero, tight, stable, purely infinite, nuclear, separable C^*-algebras over X with all simple subquotients in the bootstrap class and with free K_1-groups?
- Assume X has the unique path property. Do $\mathrm{FK}_{\mathcal{ST}}$ and FK coincide?

It is also unresolved whether $\mathrm{FK}_{\mathcal{ST}}$ and FK coincide for general finite T_0-space X, but even though a negative answer would be unpleasant, it is not clear what to expect.

CHAPTER 2

Filtered K-theory

In this chapter, the notion of a C^*-algebra over a topological space X is defined, X-equivariant KK-theory is introduced, filtered K-theory is defined, and an overview of the known results on classification of nonsimple, purely infinite, nuclear, separable C^*-algebras using filtered K-theory is given.

2.1. C^*-algebras over X and $KK(X)$-theory

The notion of C^*-algebras over a topological space is quite useful for defining what it means for maps — on the C^*-algebraical level as well as on the K-theoretical level — to preserve or respect the ideal structure of nonsimple C^*-algebras,

Let $\mathcal{O}(X)$ denote the open subsets of X, and $\mathbb{I}(A)$ denote the lattice of (two-sided, closed) ideals in A. A *C^*-algebra over a topological space* X is a pair (A, ψ) consisting of a C^*-algebra A and a map $\psi\colon \mathcal{O}(X) \to \mathbb{I}(A)$ that preserves finite infima and arbitrary suprema. We then write $A(U)$ for $\psi(U)$. Assume that X is a finite topological space satisfying the T_0 *separation axiom*, i.e., having the property that $\overline{\{x\}} \neq \overline{\{y\}}$ for all $x, y \in X$ with $x \neq y$, where $\overline{Y}$ denotes the closure of a subset Y in X. Then a C^*-algebra over X can equivalently be defined as a pair (A, ψ^*) consisting of a C^*-algebra A and a a continuous map $\psi^*\colon \operatorname{Prim}(A) \to X$, where $\operatorname{Prim}(A)$ denotes the primitive ideal space of A.

We call the C^*-algebra A *tight* over X if the map $\psi\colon \mathcal{O}(X) \to \mathbb{I}(A)$ is a lattice isomorphism, or equivalently if the map $\psi^*\colon \operatorname{Prim}(A) \to X$ is a homeomorphism.

The *locally closed subsets* of X are denoted by $\mathbb{LC}(X) = \{U \backslash V \mid V, U \in \mathcal{O}(X), V \subseteq U\}$, and the connected, nonempty, locally closed subsets of X are denoted by $\mathbb{LC}(X)^*$. For $Y \in \mathbb{LC}(X)$ we define $A(Y) = A(U)/A(V)$ when $Y = U \backslash V$ for some $V, U \in \mathcal{O}(X)$ satisfying $V \subseteq U$. Up to natural isomorphism, $A(Y)$ does not depend on the choice of U and V.

For C^*-algebras A and B over X, we say that a $*$-homomorphism $\varphi\colon A \to B$ is *X-equivariant* if $\varphi(A(U)) \subseteq B(U)$ holds for all $U \in \mathcal{O}(X)$. An X-equivariant homotopy (φ_t) is then a homotopy with the property that φ_t is X-equivariant for all $t \in [0, 1]$. An extension $A \hookrightarrow B \twoheadrightarrow C$ is called X-equivariant if it induces an extension $A(U) \hookrightarrow B(U) \twoheadrightarrow C(U)$ for all $U \in \mathcal{O}(X)$.

E. Kirchberg has constructed X-equivariant KK-theory, $KK_*(X; -, -)$, for separable C^*-algebras over X, and equipped it with an X-equivariant Kasparov product

$$- \boxtimes -\colon KK_i(X; A, B) \otimes KK_j(X; B, C) \to KK_{i+j}(X; A, C).$$

The functor $KK_*(X;-,-)$ is covariant in the first variable and contravariant in the second, it is invariant under X-equivariant homotopies and stable isomorphisms, and has the property that the functors $KK_i(X;-,-)$, $KK_{i+1}(X;\mathrm{S}-,-)$, $KK_{i+1}(X;-,\mathrm{S}-)$, and $KK_i(X;\mathrm{S}-,\mathrm{S}-)$ are equivalent. The X-equivariant KK-theory is also called ideal related KK-theory and is here refered to as $KK(X)$-theory.

Let $\mathfrak{KK}(X)$ denote the category with objects separable C^*-algebras over X and morphism groups $KK_0(X;A,B)$. A $KK(X)$-*equivalence* between C^*-algebras A and B in $\mathfrak{KK}(X)$ is then a class α in $KK_0(X;A,B)$ for which there exists a class β in $KK_0(X;B,A)$ such that $\alpha \boxtimes \beta = \mathrm{id}_A$ and $\beta \boxtimes \alpha = \mathrm{id}_B$ in $KK_0(X;A,A)$ respectively $KK_0(X;B,B)$. In particular, X-equivariant isomorphisms induce $KK(X)$-equivalences. E. Kirchberg proved the following powerful result.

THEOREM 2.1.1 ([**Kir00**, 4.3]). *Let A and B be tight, stable, O_∞-absorbing, nuclear, separable C^*-algebras over the space X. Then any $KK(X)$-equivalence between A and B is induced by an X-equivariant isomorphism between A and B.*

Recall that there are three notions of pure infiniteness for nonsimple C^*-algebras, namely pure infiniteness, strong pure infiniteness, and O_∞-absorbtion, introduced by E. Kirchberg and M. Rørdam; cf. [**KR00**] and [**KR02**].

THEOREM 2.1.2 ([**KR02**, 9.1]). *Let A be a separable C^*-algebra. If A is O_∞-absorbing, then A is strongly purely infinite. If A is strongly purely infinite, then A is purely infinite.*

Assume furthermore that A is simple and nuclear. Then A absorbs O_∞ if and only if A is purely infinite.

For nuclear, separable C^*-algebras with a finite primitive ideal space, the three notions of pure infiniteness for nonsimple C^*-algebras coincide, i.e., a purely infinite, nuclear, separable C^*-algebra with a finite primitive ideal space will always be O_∞-absorbing. Since the simple subquotients of such a C^*-algebra are O_∞-absorbing by the above theorem, this follows from applying the following theorem by A. Toms and W. Winter finitely many times.

THEOREM 2.1.3 ([**TW07**, 4.3]). *Let $I \hookrightarrow A \twoheadrightarrow A/I$ be an extension of separable C^*-algebras. If I and A/I are O_∞-absorbing, then so is A.*

To complete the picture of the $KK(X)$-equivalence classes, R. Meyer and R. Nest have proved the following.

THEOREM 2.1.4 ([**MN09**, 5.3]). *For a finite T_0-space X, any nuclear C^*-algebra in $\mathfrak{KK}(X)$ is $KK(X)$-equivalent to a tight, stable, purely infinite, nuclear C^*-algebra in $\mathfrak{KK}(X)$.*

2.2. The Meyer-Nest method for establishing UCTs

In [**MN10**], R. Meyer and R. Nest have developed a general theory for proving a Universal Coefficient Theorem, i.e., establishing short exactness of

$$\mathrm{Ext}^1_{\mathfrak{C}}(F(A),\Sigma F(B)) \hookrightarrow \mathfrak{T}(A,B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}}(F(A),F(B)),$$

for a stable homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ and objects A and B in a suitable subcategory of $\mathfrak{T}$.

In this section, an overview of their results is given, and in the next section, their definition of filtered K-theory is given, and it is explained how they apply their results to filtered K-theory.

2.2.1. The setting. In the following, $\mathfrak{T}$ denotes a triangulated category, and $\mathfrak{C}$ denotes an abelian category equipped with a suspension Σ, i.e., an additive automorphism. The suspension automorphism in $\mathfrak{T}$ is denoted by Σ, and exact triangles in $\mathfrak{T}$

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h", "\circ"{marking}] & \end{tikzcd}$$

are written $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

A functor $F\colon \mathfrak{T} \to \mathbb{C}$ is called *homological* if $F(A) \to F(B) \to F(C)$ is exact at $F(B)$ for all exact triangles $A \to B \to C \to \Sigma A$, and it is called *stable* if it intertwines the suspensions in $\mathfrak{T}$ and $\mathfrak{C}$, i.e., if $F\Sigma = \Sigma F$.

For a stable, homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$, its *kernel* $\ker F$ is defined as the subcategory of $\mathfrak{T}$ with same class of objects as $\mathfrak{T}$ and morphisms

$$\ker F(A, B) = \{f \in \mathfrak{T}(A, B) \mid F(f) = 0\}.$$

A subcategory $\mathfrak{I}$ of $\mathfrak{T}$ is called a *homological ideal* in $\mathfrak{T}$ if it is the kernel of a stable, homological functor.

2.2.2. $\mathfrak{I}$-projective resolutions. As $\mathfrak{T}$ is not abelian, there is no notion of projective resolutions of objects in $\mathfrak{T}$. For a fixed homological ideal $\mathfrak{I}$, one can however define projective resolutions relative to $\mathfrak{I}$.

DEFINITION 2.2.1 ([**MN10**]). Homological notions relative to a homological ideal $\mathfrak{I}$ are defined in the following way:

- A stable, homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ is called $\mathfrak{I}$-*exact* if $\mathfrak{I} \subseteq \ker F$.
- An object P in $\mathfrak{T}$ is called $\mathfrak{I}$-*projective* if $\mathfrak{T}(P, -)\colon \mathfrak{T} \to \mathfrak{Ab}$ is $\mathfrak{I}$-exact.
- An object A in $\mathfrak{T}$ is called $\mathfrak{I}$-*contractible* if $\mathrm{id}_A \in \mathfrak{I}(A, A)$.
- A morphism $A \xrightarrow{f} B$ is called an $\mathfrak{I}$-*phantom map* if $f \in \mathfrak{I}(A, B)$.
- A morphism $A \xrightarrow{f} B$ is called $\mathfrak{I}$-*monic* if $h \in \mathfrak{I}(C, \Sigma A)$ when f is (uniquely) embedded in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.
- A morphism $A \xrightarrow{f} B$ is called $\mathfrak{I}$-*epic* if $g \in \mathfrak{I}(B, C)$ when f is (uniquely) embedded in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.
- A morphism $A \xrightarrow{f} B$ is called an $\mathfrak{I}$-*equivalence* if $g \in \mathfrak{I}(B, C)$ and $h \in \mathfrak{I}(C, \Sigma A)$ when f is (uniquely) embedded in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.
- An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is called $\mathfrak{I}$-*exact* if $h \in \mathfrak{I}(C, \Sigma A)$.

- A chain complex (C_n, d_n) in $\mathfrak{T}$ is called $\mathfrak{J}$*-exact* if for all $n \in \mathbb{Z}$, $X_n \overset{g_n}{\to} \Sigma C_n \overset{\Sigma f_{n+1}}{\to} \Sigma X_{n+1}$ belongs to $\mathfrak{J}(X_n, \Sigma X_{n+1})$ when d_n is (uniquely) embedded in an exact triangle $C_n \overset{d_n}{\to} C_{n-1} \overset{f_n}{\to} X_n \overset{g_n}{\to} \Sigma C_n$.

An $\mathfrak{J}$*-projective resolution* of an object A in $\mathfrak{T}$ is an $\mathfrak{J}$-exact chain complex $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to A$ with P_n $\mathfrak{J}$-projective for all $n \geq 0$.

We say that there are *enough* $\mathfrak{J}$*-projective objects* in $\mathfrak{T}$, if for all A in $\mathfrak{T}$, there exists an $\mathfrak{J}$-projective object P and an $\mathfrak{J}$-epic morphism $P \to A$. If $\mathfrak{T}$ has enough $\mathfrak{J}$-projective objects, then any object in $\mathfrak{T}$ has an $\mathfrak{J}$-projective resolution by [**MN10**, 3.26].

If the triangulated category $\mathfrak{T}$ has enough $\mathfrak{J}$-projective objects, then for an object A in $\mathfrak{T}$, $\mathrm{pd}_{\mathfrak{T},\mathfrak{J}}(A)$ denotes the $\mathfrak{J}$*-projective dimension* of A in $\mathfrak{T}$, i.e., the minimal length of an $\mathfrak{J}$-projective resolution of A in $\mathfrak{T}$. Similarly, if the abelian category $\mathfrak{C}$ has enough projective objects, $\mathrm{pd}_{\mathfrak{C}}(A)$ denotes the projective dimension of the object A in $\mathfrak{C}$.

2.2.3. Universal Coefficient Theorem. An $\mathfrak{J}$-exact, stable, homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ is called *universal* if any other $\mathfrak{J}$-exact, stable, homological functor $G\colon \mathfrak{T} \to \mathfrak{C}'$ factors through it as $G = \bar{G}F$ with $\bar{G}\colon \mathfrak{C} \to \mathfrak{C}'$ a stable, exact functor unique up to natural isomorphism. The universal $\mathfrak{J}$-exact, stable, homological functor is unique up to natural isomorphism.

To establish a UCT for an $\mathfrak{J}$-exact, stable, homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$, we need to construct a one-to-one correspondance between projective resolutions in $\mathfrak{C}$ and $\mathfrak{J}$-projective resolutions in $\mathfrak{T}$. To do this, we define the *(partially defined) left adjoint of* F, denoted $F^{\vdash}$. Given an object A in $\mathfrak{C}$, we consider the functor $\mathfrak{C}(A, F(-))$. If this functor is representable, i.e., if it is equivalent to the functor $\mathfrak{T}(A', -)$ for some object A' in $\mathfrak{T}$, we define $F^{\vdash}(A)$ as the representing object A'. Note that A' is unique up to equivalence in $\mathfrak{T}$. The left adjoint functor $F^{\vdash}$ may not be defined on all of $\mathfrak{C}$ but only on a full subcategory.

Note that for any object A, if $F^{\vdash}(A)$ is defined, then it is an $\ker F$-projective object as the functor $\mathfrak{C}(A, F(-))$ vanishes on $\ker F$.

The following theorem is a consequence of [**MN10**, 3.41] which says that under the stated assumptions, F and $F^{\vdash}$ give an equivalence of categories between the full subcategory of $\mathfrak{J}$-projective objects in $\mathfrak{T}$ and the full subcategory of projective objects in $\mathfrak{C}$, in such a way that an object A in $\mathfrak{T}$ is $\mathfrak{J}$-projective if and only if $F(A)$ is projective and $\mathfrak{C}(F(A), F(B)) \cong \mathfrak{T}(A, B)$ for all objects B in $\mathfrak{T}$, and that for an object A in $\mathfrak{T}$, the functors F and $F^{\vdash}$ induce bijections between isomorphism classes of $\mathfrak{J}$-projective resolutions of A in $\mathfrak{T}$ and isomorphism classes of projective resolutions of $F(A)$ in $\mathfrak{C}$, so in particular $\mathrm{pd}_{\mathfrak{T},\mathfrak{J}}(A) = \mathrm{pd}_{\mathfrak{C}}(F(A))$.

Theorem 2.2.2 ([**MN10**, 3.41]). *Let* $\mathfrak{J}$ *be a homological ideal in the triangulated category* $\mathfrak{T}$*, let* $\mathfrak{C}$ *be a graded abelian category, and let* $F\colon \mathfrak{T} \to \mathfrak{C}$ *be an* $\mathfrak{J}$*-exact, stable, homological functor. Assume that idempotents in* $\mathfrak{T}$ *split, and assume that* F *is the universal* $\mathfrak{J}$*-exact functor and that* $\mathfrak{T}$ *has enough* $\mathfrak{J}$*-projective objects.*

Then F induces an equivalence between the bifunctors $\mathrm{Ext}^n_{\mathfrak{T},\mathfrak{J}}(-,-)$ *and* $\mathrm{Ext}^n_{\mathfrak{C}}(F(-),F(-))$ *for all* $n \geq 0$.

Combining the theorem above with the following theorem, a Universal Coefficient Theorem is established.

Theorem 2.2.3 ([**MN10**, 4.4]). *Let $\mathfrak{J}$ be a homological ideal in $\mathfrak{T}$, let A and B be objects in $\mathfrak{T}$, and assume that A has an $\mathfrak{J}$-projective resolution of length 1 and that $\mathfrak{T}(A,C)=0$ for all $\mathfrak{J}$-contractible objects C in $\mathfrak{T}$. Then*

$$\mathrm{Ext}^1_{\mathfrak{T},\mathfrak{J}}(\Sigma A,B) \hookrightarrow \mathfrak{T}(A,B) \twoheadrightarrow \mathrm{Ext}^0_{\mathfrak{T},\mathfrak{J}}(A,B)$$

is short exact.

Remark 2.2.4. To establish one of the needed assumptions for the above theorems, R. Meyer and R. Nest note the following in [**MN10**, 3.37]. If there exists a finite family of $\mathfrak{J}$-exact, stable, homological functors $F_i\colon \mathfrak{T} \to \mathfrak{C}_i$, $i \in I$, having the properties that

- for all $i \in I$, the left adjoint functor $F_i^{\vdash}$ is defined on the projective objects in $\mathfrak{C}_i$
- for all $i \in I$ and all objects A in $\mathfrak{T}$, there exists a surjection $P \twoheadrightarrow F_i(A)$ with P a projective object in $\mathfrak{C}_i$

then $\mathfrak{T}$ has enough $\mathfrak{J}$-projective objects and these are generated by

$$\bigcup_{i\in I}\{F_i^{\vdash}(P) \mid P \text{ is a projective object in } \mathfrak{C}_i\}.$$

2.3. Universal Coefficient Theorem for filtered K-theory

In [**MN09**], R. Meyer and R. Nest show that the category $\mathfrak{KK}(X)$ becomes a triangulated category when equipped with usual suspension $C_0(\mathbb{R})\otimes -$, denoted $\mathrm{S}-$, and with mapping cone sequences as exact triangles.

Let X be a finite T_0-space. For a C^*-algebra A over X and a $Y \in \mathbb{LC}(X)$, the functor $\mathrm{FK}_Y\colon \mathfrak{KK}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$ is defined as $\mathrm{FK}_Y(A) = K_*(A(Y))$. Write $\mathrm{FK}^i_Y(A)$ for $K_i(A(Y))$.

In [**MN**], R. Meyer and R. Nest construct for a fixed finite T_0-space X and for each $Y \in \mathbb{LC}^*(X)$, commutative, separable C^*-algebras R_Y over X that represent the functor $\mathrm{FK}_Y\colon \mathfrak{KK}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$, i.e., such that the functors $KK_*(X;R_Y,-)$ and FK_Y are equivalent. The representing objects are constructed such that there are extensions $R_{Y\setminus U} \hookrightarrow R_Y \twoheadrightarrow R_U$ when $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$. The representing objects R_Y will be described in Section 2.7.

2.3.1. Natural transformations between FK_Y and FK_Z.

Filtered K-theory should consist of the functors FK_Y together with natural transformations between them, i.e., $\beta^Z_Y\colon \mathrm{FK}_Y \to \mathrm{FK}_Z$ that satisfy

$$\begin{tikzcd}
\mathrm{FK}_Y(A) \arrow[r, "\beta^Z_Y(A)"] \arrow[d, "\mathrm{FK}_Y(\alpha)"'] & \mathrm{FK}_Z(A) \arrow[d, "\mathrm{FK}_Z(\alpha)"] \\
\mathrm{FK}_Y(B) \arrow[r, "\beta^Z_Y(B)"'] & \mathrm{FK}_Z(B)
\end{tikzcd}$$

for all $A, B \in \mathfrak{KK}(X)$ and all $\alpha \in KK_*(X; A, B)$. Examples of natural transformations between these functors are *extension maps* i_U^Y, *restriction maps* $r_Y^{Y\backslash U}$, and *boundary maps* $\delta_{Y\backslash U}^U$ for $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$ appearing in the six-term exact sequence

$$\begin{tikzcd}
\mathrm{FK}_U(A) \arrow[rr, "i_U^Y"] & & \mathrm{FK}_Y(A) \arrow[dl, "r_Y^{Y\backslash U}"] \\
& \mathrm{FK}_{Y\backslash U}(A) \arrow[ul, "\circ" marking, "\delta_{Y\backslash U}^U"] &
\end{tikzcd}$$

induced by $A(U) \hookrightarrow A(Y) \twoheadrightarrow A(Y\backslash U)$.

In order for filtered K-theory FK to be the universal $\ker \mathrm{FK}$-exact functor (cf. Section 2.3.5) all natural transformations between all FK_Y and FK_Z must be included. Since the functors FK_Y are representable, the Yoneda Lemma determines all natural transformations between them.

The Yoneda Lemma ([**ML98**, 3.2]). Let D be a category with small hom-sets and let r, s be objects in D. Then there is a bijection between $D(s, r)$ and the natural transformations from $D(r, -)$ to $D(s, -)$. The bijection is given by $h \mapsto D(h, -)$.

By the Yoneda Lemma, the set $\mathcal{NT}(Y, Z)$ of all natural transformations from the functor FK_Y to the functor FK_Z is given by $KK_*(X; R_Z, R_Y)$. Given $\alpha \in KK_*(X; R_Z, R_Y)$ we denote by $\bar{\alpha}$ the corresponding element in $\mathcal{NT}(Y, Z)$ given by $\alpha \boxtimes -$ where $- \boxtimes -$ denotes the Kasparov product. Given $f \in \mathcal{NT}(Y, Z)$, we let $\hat{f}$ denote the corresponding element in $KK_*(X; R_Z, R_Y)$. Let $\mathcal{NT}_i(Y, Z)$ denote the subgroup corresponding to $KK_i(X; R_Z, R_Y)$.

The extension map i_U^Y, the restriction map $r_Y^{Y\backslash U}$, and the boundary map $\delta_{Y\backslash U}^U$ for $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$ correspond to the $KK(X)$-classes represented by $R_Y \twoheadrightarrow R_U$, $R_{Y\backslash U} \hookrightarrow R_Y$, and $R_{Y\backslash U} \hookrightarrow R_Y \twoheadrightarrow R_U$, respectively, by [**MN**, 2.19].

For all finite T_0-spaces X where $\bigoplus_{Y,Z\in\mathbb{LC}(X)} \mathcal{NT}(Y, Z)$ has been calculated, it is generated by extension maps, restriction maps, and boundary maps; cf. [**Ben10**]. It is unknown whether this holds in general.

2.3.2. The target category $\mathrm{Mod}(\mathcal{NT})_c$**.** Let $\mathcal{NT}$ denote the category with objects $\mathbb{LC}(X)$ and morphism groups $\mathcal{NT}(Y, Z)$. Let $\mathrm{Mod}(\mathcal{NT})$ denote the *category of modules over* $\mathcal{NT}$, i.e., grading preserving, additive functors $G\colon \mathcal{NT} \to \mathfrak{Ab}^{\mathbb{Z}/2}$. Hence an $\mathcal{NT}$*-module* M consists of pairs of abelian groups $M(Y) = (M(Y)_0, M(Y)_1)$, for all $Y \in \mathbb{LC}(X)$, and product maps

$$M(Y)_i \times \mathcal{NT}_j(Y, Z) \to M(Z)_{i+j}$$

that are associative and additive in each variable, and satify that $\mathrm{id}_{R_Y} \in \mathcal{NT}(Y, Y)$ acts as the identity on $M(Y)$.

Equivalently, as $\bigoplus \mathcal{NT}(Y, Z)$ is a unital ring, $\mathrm{Mod}(\mathcal{NT})$ is equivalent to the right modules over $\bigoplus \mathcal{NT}(Y, Z)$. Therefore, $\mathrm{Mod}(\mathcal{NT})$ is an abelian category with enough projective objects.

Denote by $\mathrm{Mod}(\mathcal{NT})_c$ the full subcategory of $\mathrm{Mod}(\mathcal{NT})$ whose class of objects are those M for which the group $M(Y)$ is countably generated for all $Y \in \mathbb{LC}(X)$.

Definition 2.3.1 ([**MN**, 2.4]). For a fixed finite T_0-space X, the functor $\mathrm{FK}\colon \mathfrak{KK}(X) \to \mathrm{Mod}(\mathcal{NT})_c$ is defined as $\mathrm{FK}(A)(Y) = \mathrm{FK}_Y(A)$ and

$$\mathrm{FK}_Y(A) \times \mathcal{NT}(Y, Z) \to \mathrm{FK}_Z(A)$$

induced by the Kasparov product. We call $\mathrm{FK}(A)$ the *filtered K-theory* of the C^*-algebra A over X.

An $\mathcal{NT}$-module M is called *exact* if the sequence

$$\begin{tikzcd} M(U) \arrow[rr, "i_U^Y"] & & M(Y) \arrow[dl, "r_Y^{Y\setminus U}"] \\ & M(Y\setminus U) \arrow[ul, "\delta_{Y\setminus U}^U", "\circ"{marking}] & \end{tikzcd}$$

is exact for all $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$. Clearly, $\mathrm{FK}(A)$ is an exact $\mathcal{NT}$-module for any C^*-algebra A over X. For general X, it is unknown whether all exact $\mathcal{NT}$-modules arise as the filtered K-theory of a C^*-algebra over X.

In [**ABK**], the category $\mathcal{ST}$ with objects $\mathbb{LC}(X)$ and morphisms generated by extension, restriction, and boundary maps is introduced. Recall that it is unknown whether $\mathcal{ST}$ and $\mathcal{NT}$ coincide.

Definition 2.3.2. For a finite T_0-space X, the functor $\mathrm{FK}_{\mathcal{ST}}\colon \mathfrak{KK}(X) \to \mathrm{Mod}(\mathcal{ST})$ is defined as $\mathrm{FK}_{\mathcal{ST}}(A)(Y) = \mathrm{FK}_Y(A)$ and

$$\mathrm{FK}_Y(A) \times \mathcal{ST}(Y, Z) \to \mathrm{FK}_Z(A)$$

induced by the Kasparov product. We call $\mathrm{FK}_{\mathcal{ST}}(A)$ the *concrete filtered K-theory* of the C^*-algebra A over X.

Note that concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$ is the invariant one intuitively wants to define, while the abstract definition of filtered K-theory FK is needed to establish a Universal Coefficient Theorem.

2.3.3. The bootstrap class $\mathcal{B}(X)$. In [**MN09**], R. Meyer and R. Nest define for a fixed finite T_0-space X, the *bootstrap class* $\mathcal{B}(X)$ as the localising subcategory of $\mathfrak{KK}(X)$ generated by $\{i_x(\mathbb{C}) \mid x \in X\}$, where $i_x(\mathbb{C})$ denotes the C^*-algebra A over X defined by $A(U) = \mathbb{C}$ when $x \in U$ and $A(U) = 0$ when $x \notin U$.

In [**MN09**, 4.13], R. Meyer and R. Nest show that for a nuclear C^*-algebra A over X, the C^*-algebra A belongs to the bootstrap class $\mathcal{B}(X)$ if and only if $A(x)$ belongs to the bootstrap class of J. Rosenberg and C. Schochet for all $x \in X$.

R. Meyer and R. Nest give the following two characterizations of $\mathcal{B}(X)$.

Proposition 2.3.3 ([**MN09**, 4.17, 4.18]). *A C^*-algebra A over X belongs to the bootstrap class $\mathcal{B}(X)$ if and only if $KK_*(X; A, B)$ vanishes for all B in $\mathfrak{KK}(X)$ for which* $\mathrm{FK}(B) = 0$.

PROPOSITION 2.3.4 ([**MN**, 4.6]). *A separable C^*-algebra A over X belongs to the bootstrap class $\mathcal{B}(X)$ if and only if it is $KK(X)$-equivalent to a tight, stable, purely infinite, nuclear, separable C^*-algebra B over X satisfying the property that $B(x)$ belongs to the bootstrap class of J. Rosenberg and C. Schochet for all $x \in X$.*

2.3.4. Universal Coefficient Theorem for filtered K-theory. To apply the machinery of R. Meyer and R. Nest in Section 2.2.3, three things should be done: identify the objects A in $\mathfrak{KK}(X)$ for which $KK_*(X; A, B)$ vanishes for all $\ker \mathrm{FK}$-contractible objects B in $\mathfrak{KK}(X)$, establish the functor FK as the universal $\ker \mathrm{FK}$-exact functor, and show that the category $\mathfrak{KK}(X)$ has enough $\ker \mathrm{FK}$-projective objects.

By the characterization in Proposition 2.3.3 of the bootstrap class $\mathcal{B}(X)$, the objects A in $\mathfrak{KK}(X)$ for which $KK_*(X; A, B)$ vanishes for all $\ker \mathrm{FK}$-contractible objects B in $\mathfrak{KK}(X)$, are exactly the objects in $\mathcal{B}(X)$.

Using that $\mathcal{NT}(Y, Z)$ denotes *all* natural transformations from FK_Y to FK_Z, R. Meyer and R. Nest prove that FK is the universal $\ker \mathrm{FK}$-exact functor.

THEOREM 2.3.5 ([**MN**, 4.7]). *The functor* $\mathrm{FK}\colon \mathfrak{KK}(X) \to \mathrm{Mod}(\mathcal{NT})_c$ *is the universal* $\ker \mathrm{FK}$-*exact, stable, homological functor.*

In [**MN**, 4.4, 4.5], R. Meyer and R. Nest show that $\mathfrak{KK}(X)$ has enough $\ker \mathrm{FK}$-projective objects. For each $\mathrm{FK}_Y\colon \mathfrak{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2}$ they note that $\mathrm{FK}_Y^{\vdash}(\mathbb{Z}[0]) = R_Y$ as $\mathrm{Hom}(\mathbb{Z}[0], \mathrm{FK}_Y(-))$ is equivalent to $KK_*(X; R_Y, -)$, and $\mathrm{FK}_Y^{\vdash}(\mathbb{Z}[1]) = \mathrm{S}R_Y$ then follows. Hence by additivity the adjoint functor $\mathrm{FK}_Y^{\vdash}$ is defined on all pairs of free abelian groups, i.e., all projective objects in $\mathfrak{Ab}^{\mathbb{Z}/2}$, so by Remark 2.2.4, there are enough $\ker \mathrm{FK}$-projective objects in $\mathfrak{KK}(X)$.

Hence by the results in Section 2.2.3, the following may be concluded.

THEOREM 2.3.6 ([**MN**, 4.8]). *Let A and B be C^*-algebras in $\mathfrak{KK}(X)$, assume that A belongs to the bootstrap class $\mathcal{B}(X)$, and assume that* $\mathrm{FK}(A)$ *has projective dimension* $\mathrm{pd}_{\mathcal{NT}} \mathrm{FK}(A)$ *at most 1 in* $\mathrm{Mod}(\mathcal{NT})_c$.

Then the sequence

$$\mathrm{Ext}^1_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \overset{\iota}{\hookrightarrow} KK_*(X; A, B) \overset{\pi}{\twoheadrightarrow} \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)),$$

where ι is odd and π even and induced by the Kasparov product, is short exact.

COROLLARY 2.3.7 ([**MN**, 4.9]). *Let A and B be C^*-algebras over X belonging to the bootstrap class $\mathcal{B}(X)$, and assume that* $\mathrm{FK}(A)$ *and* $\mathrm{FK}(B)$ *have projective dimension at most 1 in* $\mathrm{Mod}(\mathcal{NT})_c$.

Then any morphism $\mathrm{FK}(A) \to \mathrm{FK}(B)$ *in* $\mathrm{Mod}(\mathcal{NT})_c$ *lifts to an element in $KK_0(X; A, B)$, and any isomorphism* $\mathrm{FK}(A) \to \mathrm{FK}(B)$ *in* $\mathrm{Mod}(\mathcal{NT})_c$ *lifts to a $KK(X)$-equivalence.*

2.4. Projective dimension of $\mathrm{FK}(A)$ in $\mathrm{Mod}(\mathcal{NT})_c$

A new question now arises: does $\mathrm{pd}_{\mathcal{NT}} \mathrm{FK}(A) \leq 1$ hold for all C^*-algebras A in $\mathcal{B}(X)$ for all finite T_0-spaces X?

In order to describe the spaces X for which $\operatorname{pd}_{\mathcal{NT}} \operatorname{FK}(-) \leq 1$ holds, we define a partial order on the finite T_0-space X the following way: $x \leq y$ when $\overline{\{x\}} \subseteq \overline{\{y\}}$. As X satisfies the T_0 separation axiom, this partial ordering completely determines the topology on X.

As X is finite, $(X, \leq)$ can be represented by a finite directed graph with vertices elements in X and an edge from x to y if and only if $x > y$ and $x > z \geq y$ implies $z = y$.

In [**MN**], R. Meyer and R. Nest show that if X is linear, then $\operatorname{pd}_{\mathcal{NT}} \operatorname{FK}(A)$ is at most 1 for all C^*-algebras A in $\mathfrak{KK}(X)$. The space X is called *linear* if $(X, \leq)$ is totally ordered. A tight C^*-algebra A over a linear space $X = \{x_1, \ldots, x_n\}$ with $x_i \leq x_j$ when $i \geq j$ is then a C^*-algebra with linear ideal lattice

$$0 \subsetneq A(x_1) \subsetneq A(\{x_1, x_2\}) \subsetneq \cdots \subsetneq A(\{x_1, \ldots, x_{n-1}\}) \subsetneq A.$$

Using their methods, R. Bentmann shows in [**Ben10**] that if X is an accordion space, then $\operatorname{pd}_{\mathcal{NT}} \operatorname{FK}(A) \leq 1$ holds for all C^*-algebras A in $\mathfrak{KK}(X)$. The space X is called an *accordion space* if it is connected, all vertices in its representing graph have unoriented degree at most 2, i.e., at most two ingoing or outgoing edges, and exactly two vertices in its representing graph have unoriented degree 1. So, an accordion space is a space whose representing graph looks like an accordion. A linear space is an accordion space, all spaces with at most 3 points are accordion spaces, and in Section 2.5, examples of four-point spaces that are not accordion spaces will be given.

The projective dimension of $\operatorname{FK}(A)$ is mainly a question of properties of $\operatorname{Mod}(\mathcal{NT})_c$. One can show that projective modules in $\operatorname{Mod}(\mathcal{NT})_c$ are exact and have free entries. For X linear, R. Meyer and R. Nest show that all exact $\mathcal{NT}$-modules with free entries are projective, and using this they prove that all exact modules in $\operatorname{Mod}(\mathcal{NT})_c$ have projective dimension at most 1. For accordion spaces, R. Bentmann establish the same properties.

Furthermore, R. Meyer and R. Nest, and R. Bentmann show that for linear spaces and the more general accordion spaces, all exact objects in $\operatorname{Mod}(\mathcal{NT})_c$ arise as the filtered K-theory of a C^*-algebra.

In [**MN**], R. Meyer and R. Nest consider the four-point space $\mathcal{W}$, which will be defined in Section 2.5, and construct a C^*-algebra A in $\mathcal{B}(\mathcal{W})$ satisfying $\operatorname{pd}_{\mathcal{NT}} \operatorname{FK}(A) = 2$. Using this C^*-algebra A, they construct non-$KK(\mathcal{W})$-equivalent C^*-algebras in $\mathcal{B}(\mathcal{W})$ that have isomorphic filtered K-theory; cf. Section 2.5. Using their methods, R. Bentmann and M. Köhler show the following.

Theorem 2.4.1 ([**BK**]). *Let X be a finite connected T_0-space. Then the following are equivalent.*

- *X is an accordion space.*
- *For all C^*-algebras A in $\mathfrak{KK}(X)$, $\operatorname{pd}_{\mathcal{NT}} \operatorname{FK}(A) \leq 1$ holds.*
- *For all A and B in $\mathcal{B}(X)$, if $\operatorname{FK}(A)$ and $\operatorname{FK}(B)$ are isomorphic, then A and B are $KK(X)$-equivalent.*

Because of the result of R. Bentmann and M. Köhler, it appears that filtered K-theory is useless for classifying C^*-algebras with primitive ideal spaces that are *nonaccordion*, i.e., are connected but are not accordion spaces. However, as we will see in Section 2.6, the situation is not hopeless.

2.5. Some counterexamples to classification

In this section the focus is on four spaces $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{D}$, and $\mathcal{S}$ and what the counterexamples to classification, constructed by R. Meyer and R. Nest, and R. Bentmann, look like for these spaces. Together with the spaces $\mathcal{W}^{\mathrm{op}}$ and $\mathcal{Y}^{\mathrm{op}}$, these spaces make up all the four-point nonaccordion spaces.

2.5.1. The counterexample over $\mathcal{W}$. The space $\mathcal{W}$ considered by R. Meyer and R. Nest in [**MN**] is defined as $\mathcal{W} = \{1,2,3,4\}$ with $\mathcal{O}(\mathcal{W}) = \{U \subseteq \mathcal{W} \mid 4 \in U\} \cup \{\emptyset\}$. The representing graph for $\mathcal{W}$ is then

$$\begin{tikzcd}
1 & 2 & 3 \\
& 4 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and a C^*-algebra A over $\mathcal{W}$ then corresponds to an extension $A(4) \hookrightarrow A \twoheadrightarrow A(1) \oplus A(2) \oplus A(3)$. To ease notation, subsets of $\mathcal{W}$ will be written, e.g., 124 for $\{1,2,4\}$.

R. Meyer and R. Nest calculated for each pair $Y, Z \in \mathbb{LC}^*(\mathcal{W})$ the group $\mathcal{NT}(Y,Z)$ using that $\mathcal{NT}(Y,Z) \cong KK_*(X; R_Z, R_Y) \cong \mathrm{FK}_Z(R_Y)$. R. Bentmann provides in [**Ben10**] a large collection of lemmas that make it possible to identify generators and establish relations in $\mathcal{NT}$. It suffices to calculate $\mathcal{NT}(Y,Z)$ for Y and Z connected since $\mathrm{FK}_Y = \mathrm{FK}_{Y_1} \oplus \cdots \oplus \mathrm{FK}_{Y_n}$ when $Y_1, \ldots, Y_n$ are the connected components of a locally closed subset Y. It turned out that for $\mathcal{W}$, the morphisms in the category are generated by the 18 natural transformations in the diagram

$$\begin{tikzcd}
& 14 \arrow[r, "i"] \arrow[dr, "i"] & 124 \arrow[dr, "i"] & & 1 \arrow[dr, "\delta", "\circ" marking] & \\
4 \arrow[ur, "i"] \arrow[r, "i"] \arrow[dr, "i"] & 24 \arrow[ur, "i"] \arrow[dr, "i"] & 134 \arrow[r, "i"] & 1234 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "r"] & 2 \arrow[r, "\delta", "\circ" marking] & 4 \\
& 34 \arrow[ur, "i"] \arrow[r, "i"] & 234 \arrow[ur, "i"] & & 3 \arrow[ur, "\delta", "\circ" marking] &
\end{tikzcd}$$

subject to the following relations:

- All six squares commute.
- For all $j \in \{1,2,3\}$, the composition of $1234\backslash j \to 1234$ with $1234 \to j$ vanishes.
- For all $j \in \{1,2,3\}$, the composition of $j \to 4$ with $4 \to j4$ vanishes.
- The sum of the three compositions $1234 \to j \to 4$, for $j \in \{1,2,3\}$, vanishes.

Notice that all the generating transformations are either extension maps, restrictions maps, or boundary maps. In this case, $\mathcal{ST}$ equals $\mathcal{NT}$.

An $\mathcal{NT}$-module is therefore determined by 11 pairs of groups, 15 even maps, and 3 odd maps. As an example, the maps appearing in the six-term sequence induced by the extension $A(4) \hookrightarrow A(134) \twoheadrightarrow A(13)$ are $i_4^{134} = i_4^{14} i_{14}^{134} = i_4^{34} i_{34}^{134}$, $r_{134}^{13} = i_{134}^{1234} r_{1234}^{1} i_1^{13} + i_{134}^{1234} r_{1234}^{3} i_3^{13}$, and $\delta_{13}^{4} = r_{13}^{1} \delta_1^{4} + r_{13}^{3} \delta_3^{4}$, as 13 is not connected.

Define for each $Y \in \mathbb{LC}^*(X)$ an $\mathcal{NT}$-module P_Y by $P_Y(Z) = \mathcal{NT}(Y, Z)$ and

$$P_Y(Z) \times \mathcal{NT}(Z, W) \to P_Y(W)$$

by composition. As an $\mathcal{NT}$-module, P_Y is free and generated by $\mathrm{id}_{R_Y} \in P_Y(Y)$, hence it is projective. Notice that $P_Y \cong \mathrm{FK}(R_Y)$.

Now, consider the injective map $P_{1234} \to P_{124} \oplus P_{134} \oplus P_{234}$ given by $\mathrm{id}_{R_{1234}} \mapsto (i_{124}^{1234}, i_{134}^{1234}, i_{234}^{1234})$ and extended by $\mathcal{NT}$-linearity, and let M denote the cokernel. Then the cokernel M has free entries and is exact but has projective dimension 1, according to [**MN**]. Let $k \geq 2$ and put $M_k = M \otimes \mathbb{Z}/k$. Then M_k is

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[dr] & \mathbb{Z}/k \arrow[dr] & & \mathbb{Z}/k \arrow[dr, "\circ" marking] & \\
\mathbb{Z}/k[1] \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[ur] \arrow[dr] & \mathbb{Z}/k \arrow[r] & (\mathbb{Z}/k)^2 \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z}/k \arrow[r, "\circ" marking] & \mathbb{Z}/k[1] \\
 & 0 \arrow[r] \arrow[ur] & \mathbb{Z}/k \arrow[ur] & & \mathbb{Z}/k \arrow[ur, "\circ" marking] &
\end{tikzcd}$$

and has projective dimension 2, and

$$0 \to P_{1234} \to P_{1234} \oplus P_{124} \oplus P_{134} \oplus P_{234} \to P_{124} \oplus P_{134} \oplus P_{234} \to M_k \to 0$$

is a projective resolution of M_k, according to [**MN**], and R. Meyer and R. Nest construct a C^*-algebra A_k in $\mathcal{B}(\mathcal{W})$ with $\mathrm{FK}(A_k) = M_k$.

Using the projective resolution of M_k, R. Meyer and R. Nest construct non-$KK(\mathcal{W})$-equivalent C^*-algebras in $\mathcal{B}(\mathcal{W})$ with filtered K-theory $M_k \oplus P_{1234}[1]$, where $P_{1234}[1]$ is

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r] \arrow[dr] & 0 \arrow[dr] & & \mathbb{Z}[1] \arrow[dr, "\circ" marking] & \\
\mathbb{Z}^2 \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z} \arrow[ur] \arrow[dr] & 0 \arrow[r] & \mathbb{Z}[1] \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z}[1] \arrow[r, "\circ" marking] & \mathbb{Z}^2 \\
 & \mathbb{Z} \arrow[r] \arrow[ur] & 0 \arrow[ur] & & \mathbb{Z}[1] \arrow[ur, "\circ" marking] & .
\end{tikzcd}$$

Please notice that for $M_k \oplus P_{1234}[1]$ the boundary maps $j \to 4$, for all $j \in \{1, 2, 3\}$, vanish on neither $M_k(j)_0 \oplus P_{1234}[1](j)_0$ nor $M_k(j)_1 \oplus P_{1234}[1](j)_1$. This implies that the non-$KK(\mathcal{W})$-equivalent C^*-algebras with this filtered K-theory do not have real rank zero, and neither do their suspensions.

2.5.2. The refined invariant FK′ over $\mathcal{W}$.

For $\mathrm{Mod}(\mathcal{NT})_c$ over $\mathcal{W}$, the problem occurs because there are too few projective objects. Note that in order to be able to construct their counterexample, R. Meyer and R. Nest used the existense of a free and exact module M that was not projective.

In an attempt to solve the problem, they add a C^*-algebra R_{12344} over $\mathcal{W}$ with $\mathrm{FK}(R_{12344}) = M$ to the class of representing objects. I.e., they define a new category $\mathcal{NT}'$ with objects $\mathbb{LC}(\mathcal{W}) \cup \{12344\}$ and morphisms $KK_*(\mathcal{W}; R_Z, R_Y)$, and define a *refined filtered K-theory* $\mathrm{FK}' \colon \mathfrak{KK}(\mathcal{W}) \to \mathrm{Mod}(\mathcal{NT}')_c$ as $\mathrm{FK}'(A)(Y) = KK_*(\mathcal{W}; R_Y, A)$ for $Y \in \mathbb{LC}(\mathcal{W}) \cup \{12344\}$, thus adding another K-group and natural transformations to and from it.

The C^*-algebra R_{12344} is defined as the mapping cone of one of the generators of the cyclic group $\mathcal{NT}(234, 14)$. Notice that R_{12344} is unique in $\mathcal{B}(\mathcal{W})$ up to $KK(\mathcal{W})$-equivalence by the UCT for filtered K-theory FK since M has projective dimension 1. The choice of generator does not affect the mapping cone, up to $KK(\mathcal{W})$-equivalence, and a generator of $\mathcal{NT}(134, 24)$ or $\mathcal{NT}(124, 34)$ will also give the same mapping cone, up to $KK(\mathcal{W})$-equivalence.

Luckily, it turns out that for all C^*-algebras A in $\mathfrak{KK}(\mathcal{W})$, $\mathrm{FK}'(A)$ has projective dimension at most 1 in $\mathrm{Mod}(\mathcal{NT}_c)$, and R. Meyer and R. Nest establish a UCT for this refined filtered K-theory.

Theorem 2.5.1 ([**MN**, 5.14]). *Let A and B be C^*-algebras in $\mathfrak{KK}(\mathcal{W})$, and assume that A belongs to the bootstrap class $\mathcal{B}(\mathcal{W})$.*

Then the sequence

$$\mathrm{Ext}^1_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)) \overset{\iota}{\hookrightarrow} KK_*(\mathcal{W}; A, B) \overset{\pi}{\twoheadrightarrow} \mathrm{Hom}_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)),$$

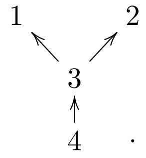
where ι is odd and π even and induced by the Kasparov product, is short exact.

If B also belongs to $\mathcal{B}(\mathcal{W})$, then any morphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to an element in $KK_0(\mathcal{W}; A, B)$, and any isomorphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to a $KK(\mathcal{W})$-equivalence.

The group $\mathrm{FK}_{12344}(A)$ is the K-theory of the pullback of $(A(124), A(234))$ along (r^2_{124}, r^2_{234}).

2.5.3. The counterexample over $\mathcal{Y}$.

In [**Ben10**], R. Bentmann considers the space $\mathcal{Y}$ defined as $\mathcal{Y} = \{1, 2, 3, 4\}$ with open subsets $\mathcal{O}(\mathcal{Y}) = \{\emptyset, 4, 34, 134, 124, 1234\}$. The representing graph of $\mathcal{Y}$ is



.

R. Bentmann calculated the morphism groups in $\mathcal{NT}$ over $\mathcal{Y}$ and discovered that they are generated by the 18 morphisms fitting into the following diagram

$$\begin{tikzcd}
 & 134 \arrow[r, "r"] \arrow[dr, "i"] & 13 \arrow[dr, "i"] & & 1 \arrow[dr, "\delta", "\circ" marking] & \\
34 \arrow[ur, "i"] \arrow[r, "r"] \arrow[dr, "i"] & 3 \arrow[ur, "i"] \arrow[dr, "i"] & 1234 \arrow[r, "r"] & 123 \arrow[ur, "r"] \arrow[r, "\delta", "\circ" marking] \arrow[dr, "r"] & 4 \arrow[r, "i"] & 34 \\
 & 234 \arrow[ur, "i"] \arrow[r, "r"] & 23 \arrow[ur, "i"] & & 2 \arrow[ur, "\delta", "\circ" marking] &
\end{tikzcd}$$

subject to the corresponding relations as for $\mathcal{W}$.

Following the same procedure as R. Meyer and R. Nest, he constructs an exact object M, namely

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[dr] & \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dr, "\circ" marking] & \\
\mathbb{Z}[1] \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[ur] \arrow[dr] & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r, "\circ" marking] \arrow[ur] \arrow[dr] & \mathbb{Z}[1] \arrow[r] & \mathbb{Z}[1] \\
 & 0 \arrow[r] \arrow[ur] & \mathbb{Z} \arrow[ur] & & \mathbb{Z} \arrow[ur, "\circ" marking] &
\end{tikzcd},$$

with projective dimension 1, defines $M_k = M \otimes \mathbb{Z}/k$, and shows that there exists non-$KK(\mathcal{Y})$-equivalent C^*-algebras in $\mathcal{B}(\mathcal{Y})$ with filtered K-theory $M_k \oplus P_{123}[1]$, where $P_{123}[1]$ is

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r] \arrow[dr] & 0 \arrow[dr] & & \mathbb{Z}[1] \arrow[dr, "\circ" marking] & \\
\mathbb{Z}^2 \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z} \arrow[ur] \arrow[dr] & 0 \arrow[r] & \mathbb{Z}[1] \arrow[r, "\circ" marking] \arrow[ur] \arrow[dr] & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \\
 & \mathbb{Z} \arrow[r] \arrow[ur] & 0 \arrow[ur] & & \mathbb{Z}[1] \arrow[ur, "\circ" marking] &
\end{tikzcd}.$$

Please notice that for $M_k \oplus P_{123}[1]$, the boundary maps $Y \to Z$, where $(Y, Z) \in \{(123, 4), (1, 34), (2, 34)\}$, vanish on neither $M_k(Y)_0 \oplus P_{123}[1](Y)_0$ nor $M_k(Y)_1 \oplus P_{123}[1](Y)_1$. This implies that the non-$KK(\mathcal{Y})$-equivalent C^*-algebras with this filtered K-theory do not have real rank zero, and neither do their suspensions.

2.5.4. The refined invariant FK' over $\mathcal{Y}$. Still following the strategy of R. Meyer and R. Nest, R. Bentmann then defines a C^*-algebra R_{12334} over $\mathcal{Y}$ as the mapping cone of a generator of $\mathcal{NT}(23, 134)$, shows that $\mathrm{FK}(R_{12334}) = M$, and defines a *refined filtered K-theory* $\mathrm{FK}'\colon \mathfrak{KK}(\mathcal{Y}) \to \mathrm{Mod}(\mathcal{NT}')_c$ where $\mathcal{NT}'$ has objects $\mathbb{LC}(\mathcal{Y}) \cup \{12334\}$ and morphism groups $KK_*(\mathcal{Y}; R_Z, R_Y)$.

Again, it turns out that for all C^*-algebras A in $\mathfrak{KK}(\mathcal{Y})$, $\mathrm{FK}'(A)$ has projective dimension at most 1 in $\mathrm{Mod}(\mathcal{NT}_c)$, and R. Bentmann establishes a UCT for this refined filtered K-theory.

THEOREM 2.5.2 ([**Ben10**, 6.1.22]). *Let A and B be C^*-algebras in $\mathfrak{KK}(\mathcal{Y})$, and assume that A belongs to the bootstrap class $\mathcal{B}(\mathcal{Y})$.*

Then the sequence

$$\mathrm{Ext}^1_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)) \overset{\iota}{\hookrightarrow} KK_*(\mathcal{Y}; A, B) \overset{\pi}{\twoheadrightarrow} \mathrm{Hom}_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)),$$

where ι is odd and π even and induced by the Kasparov product, is short exact.

If B also belongs to $\mathcal{B}(\mathcal{Y})$, then any morphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to an element in $KK_0(\mathcal{Y}; A, B)$, and then any isomorphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to a $KK(\mathcal{Y})$-equivalence.

The group $\mathrm{FK}_{12334}(A)$ is the K-theory of the pullback of $(A(13), A(1234))$ along (r^1_{13}, r^1_{1234}).

2.5.5. The counterexample over $\mathcal{D}$. In [**Ben10**], R. Bentmann also considers the space $\mathcal{D}$ defined as $\mathcal{D} = \{1,2,3,4\}$ with open subsets $\mathcal{O}(\mathcal{D}) = \{\emptyset, 4, 34, 24, 234, 1234\}$. The representing graph of $\mathcal{D}$ is

$$\begin{tikzcd}
 & 1 & \\
2 \arrow[ur] & & 3 \arrow[ul] \\
 & 4 \arrow[ul] \arrow[ur] & .
\end{tikzcd}$$

R. Bentmann calculated the morphism groups in $\mathcal{NT}$ over $\mathcal{D}$ and discovered that they are generated by the 18 morphisms fitting into the following diagram

$$\begin{tikzcd}
 & 12 \arrow[r, "\delta", "\circ" marking] \arrow[dr, "r"] & 34 \arrow[dr, "i"] & & 3 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "\delta", "\circ" marking] \arrow[dr, "r"] & 4 \arrow[ur, "i"] \arrow[dr, "i"] & 1 \arrow[r, "\delta", "\circ" marking] & 234 \arrow[r, "i"] \arrow[ur, "r"] \arrow[dr, "r"] & 1234 \arrow[r, "r"] & 123 \\
 & 13 \arrow[r, "\delta", "\circ" marking] \arrow[ur, "r"] & 24 \arrow[ur, "i"] & & 2 \arrow[ur, "i"] &
\end{tikzcd}$$

also subject to the corresponding relations as for $\mathcal{W}$.

Again following the same procedure as R. Meyer and R. Nest, he constructs an exact object M, namely

$$\begin{tikzcd}
 & 0 \arrow[r, "\circ" marking] \arrow[dr] & \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dr] & \\
\mathbb{Z} \arrow[ur] \arrow[r, "\circ" marking] \arrow[dr] & 0 \arrow[ur] \arrow[dr] & \mathbb{Z}[1] \arrow[r, "\circ" marking] & \mathbb{Z}^2 \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & 0 \arrow[r, "\circ" marking] \arrow[ur] & \mathbb{Z} \arrow[ur] & & \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

with projective dimension 1, defines $M_k = M \otimes \mathbb{Z}/k$, and shows that there exists non-$KK(\mathcal{D})$-equivalent C^*-algebras in $\mathcal{B}(\mathcal{D})$ with filtered K-theory $M_k \oplus P_{234}[1]$, where $P_{234}[1]$ is

$$\begin{tikzcd}
 & \mathbb{Z}[1] \arrow[r, "\circ" marking] \arrow[dr] & 0 \arrow[dr] & & \mathbb{Z}[1] \arrow[dr] & \\
\mathbb{Z}[1]^2 \arrow[ur] \arrow[r, "\circ" marking] \arrow[dr] & \mathbb{Z} \arrow[ur] \arrow[dr] & 0 \arrow[r, "\circ" marking] & \mathbb{Z}[1] \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{Z}[1] \arrow[r] & \mathbb{Z}[1]^2 \\
 & \mathbb{Z}[1] \arrow[r, "\circ" marking] \arrow[ur] & 0 \arrow[ur] & & \mathbb{Z}[1] \arrow[ur] & ,
\end{tikzcd}$$

Please notice that for $M_k \oplus P_{234}[1]$, all the boundary maps $Y \to Z$, where $(Y, Z) \in \{(123, 4), (12, 34), (13, 24), (1, 234)\}$, vanish on the even part. This implies that the non-$KK(\mathcal{D})$-equivalent C^*-algebras with this filtered K-theory can be chosen to have real rank zero.

THEOREM 2.5.3 ([**ARR**, 1.2]). *There exists tight, stable, purely infinite, nuclear, separable C^*-algebras A and B in the bootstrap class $\mathcal{B}(D)$ that are non-$KK(\mathcal{D})$-equivalent, have isomorphic filtered K-theory, and have real rank zero.*

2.5.6. The refined invariant FK' over $\mathcal{D}$. Again following the strategy of R. Meyer and R. Nest, R. Bentmann then defines a C^*-algebra $R_{4\setminus 1}$ over $\mathcal{D}$ as the mapping cone of a generator of $\mathcal{NT}(1,4)$, shows that $\mathrm{FK}(R_{4\setminus 1}) = M$, and defines a *refined filtered K-theory* $\mathrm{FK}'\colon \mathfrak{KK}(X) \to \mathrm{Mod}(\mathcal{NT}')_c$ where $\mathcal{NT}'$ has objects $\mathbb{LC}(X) \cup \{4\setminus 1\}$ and morphism groups $KK_*(\mathcal{D}, R_Z, R_Y)$.

Again, it turns out that for all C^*-algebras A over $\mathcal{D}$, $\mathrm{FK}'(A)$ has projective dimension at most 1 in $\mathrm{Mod}(\mathcal{NT}_c)$, and R. Bentmann establishes a UCT for this refined filtered K-theory.

THEOREM 2.5.4 ([**Ben10**, 6.2.14]). *Let A and B be C^*-algebras in $\mathfrak{KK}(\mathcal{D})$, and assume that A belongs to the bootstrap class $\mathcal{B}(\mathcal{D})$.*

Then the sequence

$$\mathrm{Ext}^1_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)) \overset{\iota}{\hookrightarrow} KK_*(\mathcal{D}; A, B) \overset{\pi}{\twoheadrightarrow} \mathrm{Hom}_{\mathcal{NT}'}(\mathrm{FK}'(A), \mathrm{FK}'(B)),$$

where ι is odd and π even and induced by the Kasparov product, is short exact.

If B also belongs to $\mathcal{B}(\mathcal{D})$, then any morphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to an element in $KK_0(\mathcal{D}; A, B)$, and then any isomorphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$ in $\mathrm{Mod}(\mathcal{NT}')_c$ lifts to a $KK(\mathcal{D})$-equivalence.

2.5.7. The counterexample over $\mathcal{S}$. In [**Ben10**], R. Bentmann also considers the space $\mathcal{S}$ defined as $\mathcal{S} = \{1,2,3,4\}$ with open subsets $\mathcal{O}(\mathcal{S}) = \{\emptyset, 4, 3, 34, 234, 134, 1234\}$. The representing graph of $\mathcal{S}$ is

$$\begin{tikzcd}
1 & 2 \\
3 \arrow[u] \arrow[ur] & 4 \arrow[u] \arrow[ul]
\end{tikzcd} .$$

R. Bentmann calculated the morphism groups in $\mathcal{NT}$ over $\mathcal{S}$ and discovered that they are generated by the 24 morphisms fitting into the following diagram

$$\begin{tikzcd}
 & & 13 \arrow[dr, "i"] & & & \\
3 \arrow[r, "i"] \arrow[ddr, "i"] & 134 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "i"] & 14 \arrow[ddr, "i"] & 123 \arrow[r, "r"] \arrow[ddr, "r"] & 1 \arrow[r, "\delta", "\circ"{marking}] \arrow[ddr, "\delta", "\circ"{marking}] & 3 \\
 & & 1234 \arrow[ur, "r"] \arrow[dr, "r"] & & & \\
4 \arrow[r, "i"] \arrow[uur, "i"] & 234 \arrow[ur, "i"] \arrow[r, "r"] \arrow[dr, "r"] & 23 \arrow[uur, "i"] & 124 \arrow[r, "r"] \arrow[uur, "r"] & 2 \arrow[r, "\delta", "\circ"{marking}] \arrow[uur, "\delta", "\circ"{marking}] & 4 \\
 & & 24 \arrow[ur, "i"] & & &
\end{tikzcd}$$

subject to some relations.

Again following the same procedure as R. Meyer and R. Nest, he constructs an exact object M, namely

$$\begin{tikzcd}
 & & \mathbb{Z}[1] \arrow[dr] & & & \\
\mathbb{Z}[1]^2 \arrow[r] \arrow[ddr] & \mathbb{Z}[1]^3 \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{Z}[1] \arrow[ddr] & \mathbb{Z} \arrow[r] \arrow[ddr] & \mathbb{Z} \arrow[r, "\circ" marking] \arrow[ddr, "\circ" marking] & \mathbb{Z}[1]^2 \\
 & & \mathbb{Z}[1] \arrow[ur] \arrow[dr] & & & \\
\mathbb{Z}[1]^2 \arrow[r] \arrow[uur] & \mathbb{Z}[1]^2 \arrow[r] \arrow[ur] \arrow[dr] & 0 \arrow[uur] & \mathbb{Z} \arrow[r] \arrow[uur] & \mathbb{Z}^2 \arrow[r, "\circ" marking] \arrow[uur, "\circ" marking] & \mathbb{Z}[1]^2 \\
 & & 0 \arrow[ur] & & &
\end{tikzcd}$$

with projective dimension 1, defines $M_k = M \otimes \mathbb{Z}/k$, and shows that there exists non-$KK(\mathcal{S})$-equivalent C^*-algebras A and B in $\mathcal{B}(\mathcal{S})$ with filtered K-theory $M_k \oplus P_1[1]$, where $P_1[1]$ is

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & & \\
\mathbb{Z} \arrow[r] \arrow[ddr] & \mathbb{Z} \arrow[r] \arrow[ur] \arrow[dr] & 0 \arrow[ddr] & 0 \arrow[r] \arrow[ddr] & \mathbb{Z}[1] \arrow[r, "\circ" marking] \arrow[ddr, "\circ" marking] & \mathbb{Z} \\
 & & \mathbb{Z} \arrow[ur] \arrow[dr] & & & \\
\mathbb{Z} \arrow[r] \arrow[uur] & \mathbb{Z}^2 \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{Z} \arrow[uur] & 0 \arrow[r] \arrow[uur] & 0 \arrow[r, "\circ" marking] \arrow[uur, "\circ" marking] & \mathbb{Z} \\
 & & \mathbb{Z} \arrow[ur] & & &
\end{tikzcd}.$$

One can check that the maps $M_k(1)_0 \to M_k(3)_1$ and $M_k(1)_0 \to M_k(4)_1$ are embeddings and therefore nonzero, hence A and B cannot have real rank zero. Also, one can check that the maps $P_1(1)_0 \to P_1(3)_1$ and $P_1(1)_0 \to P_1(4)_1$ are isomorphisms and therefore nonzero, hence $\mathsf{S}A$ and $\mathsf{S}B$ cannot have real rank zero either.

So the constructed non-$KK(\mathcal{S})$-equivariant C^*-algebras with isomorphic filtered K-theory do not have real rank zero. However, there is no known finite refinement of filtered K-theory over $\mathcal{S}$ that admits a Universal Coefficient Theorem.

2.6. Filtered K-theory for C^*-algebras of real rank zero

As noted in the previous section, the counterexamples constructed for the spaces $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{S}$ do not have real rank zero, while the counterexamples constructed for the space $\mathcal{D}$ do.

In [**ARR**], a classification result is achieved for $\mathcal{W}$ and $\mathcal{Y}$ by restricting to real rank zero C^*-algebras; cf. Theorem 2.1.1.

THEOREM 2.6.1 ([**ARR**, 1.1]). *Let X be homeomorphic to one of the four spaces $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{W}^{\mathrm{op}}$, and $\mathcal{Y}^{\mathrm{op}}$. Let A and B be real rank zero C^*-algebras in the bootstrap class $\mathcal{B}(X)$. Then any isomorphism* $\mathrm{FK}(A) \to \mathrm{FK}(B)$ *in* $\mathrm{Mod}(\mathcal{NT})_c$ *lifts to a $KK(X)$-equivalence.*

SKETCH OF PROOF. Consider the space $\mathcal{W}$. By Theorem 2.5.1, it suffices to show that for real rank zero C^*-algebras A and B over $\mathcal{W}$, any isomorphism $\varphi\colon \mathrm{FK}(A) \to \mathrm{FK}(B)$ extends to an isomorphism $\varphi'\colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. Note that φ should be extended to $\mathrm{FK}_{12344}(A) \to \mathrm{FK}_{12344}(B)$ in a way that respects the natural transformations to and from 12344.

For a real rank zero C^*-algebra A over $\mathcal{W}$, the refined filtered K-theory $\mathrm{FK}'(A)$ consists of the groups and maps

$$\begin{tikzcd}
& 34_1 \arrow[dr] & & 234_1 \arrow[dr] & & 3_1 \arrow[dr] & & 14_0 \arrow[dr] & & 124_0 \arrow[dr] & & 1_0 \\
4_1 \arrow[r] \arrow[ur] \arrow[dr] & 24_1 \arrow[r] & 12344_1 \arrow[r] \arrow[ur] \arrow[dr] & 134_1 \arrow[r] & 1234_1 \arrow[r] \arrow[ur] \arrow[dr] & 2_1 \arrow[r] & 4_0 \arrow[r] \arrow[ur] \arrow[dr] & 24_0 \arrow[r] & 12344_0 \arrow[r] \arrow[ur] \arrow[dr] & 134_0 \arrow[r] & 1234_0 \arrow[r] \arrow[ur] \arrow[dr] & 2_0 \\
& 14_1 \arrow[ur] & & 124_1 \arrow[ur] & & 1_1 \arrow[ur] & & 34_0 \arrow[ur] & & 234_0 \arrow[ur] & & 3_0
\end{tikzcd}$$

where Y_i denotes $\mathrm{FK}^i_Y(A)$. For each of the parts

$$\begin{tikzcd}
& Z_i \arrow[dr] & \\
Y_i \arrow[r] \arrow[ur] \arrow[dr] & Z'_i \arrow[r] & W_{i+j} \\
& Z''_i \arrow[ur] &
\end{tikzcd}$$

of the diagram, the sequence

$$\begin{tikzcd}
Y_0 \arrow[r] & Z_0 \oplus Z'_0 \oplus Z''_0 \arrow[r] & W_0 \arrow[d] \\
W_1 \arrow[u] & Z_1 \oplus Z'_1 \oplus Z''_1 \arrow[l] & Y_1 \arrow[l]
\end{tikzcd}$$

is exact, hence 12344_1 is isomorphic to the kernel of $124_1 \oplus 134_1 \oplus 234_1 \to 1234_1$, and 12344_0 is isomorphic to the cokernel of $4_0 \to 14_0 \oplus 24_0 \oplus 34_0$ as the maps $1234_0 \to 12344_1$ and $12344_0 \to 4_1$ factor through $2_0 \to 4_1$ which vanishes due to real rank zero.

Using this, φ can be extended to isomorphisms on 12344_1 and 12344_0. The constructed maps will respect the natural transformations as the induced maps on kernel respectively cokernel do. □

The result is somewhat surprising since R. Meyer and R. Nest have constructed a real rank zero C^*-algebra A in $\mathcal{B}(\mathcal{W})$ satisfying $\mathrm{pd}\,\mathrm{FK}(A) = 2$.

Using the same strategy, it is proved in [**ABK**] that for suitably nice C^*-algebras over $\mathcal{D}$, the refined filtered K-theory FK' can be recovered from an invariant coarser than filtered K-theory FK. See Theorem 3.2.4. In [**ABK**], the same strategy is used to proof that for real rank zero C^*-algebras and for suitably nice spaces, concrete filtered K-theory can be recovered from a coarser invariant. See Theorem 3.3.4.

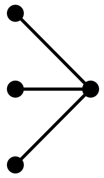
For the space $\mathcal{S}$, it is most likely that there does not exist a finite refinement FK' of FK that admits a UCT. In [**Ben10**], R. Bentmann has calculated $\mathcal{NT}$ for $\mathcal{S}$ and explains why this is unlikely. For this reason, among others, the strategy in [**ARR**] does not seem to work for general spaces.

The above results do, however, suggest that despite the counterexamples of R. Meyer and R. Nest, and R. Bentmann and M. Köhler, filtered K-theory can still turn out to be useful for classifying suitably nice C^*-algebras.

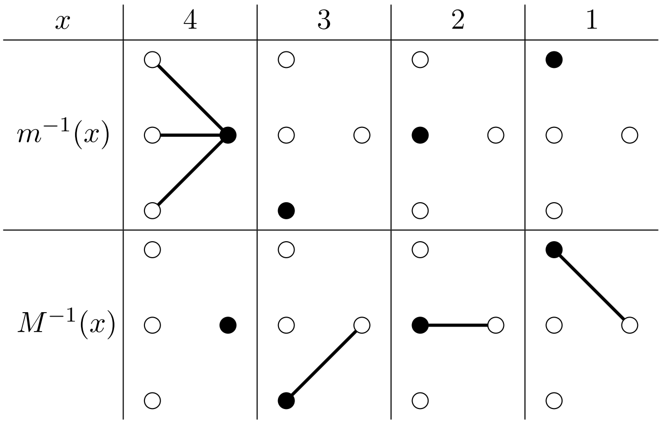
2.7. The representing objects R_Y

To give a more hands-on approach to filtered K-theory, in this last section the focus will be on the representing objects R_Y. There will not be given any details of the proof of the equivalence of functors between FK_Y and $K_*(X; R_Y, -)$ but only the definition of the objects R_Y.

The construction of R. Meyer and R. Nest in [**MN**] goes as follows. The space $\mathcal{W}$ will be used as an example. Consider the geometric realization $\mathrm{Ch}(X)$ of the nerve of X, i.e., the simplicial set whose nondegenerate n-simplices $[x_0, \dots, x_n]$ are strict chains $x_0 < \cdots < x_n$. For the space $\mathcal{W}$, $\mathrm{Ch}(\mathcal{W})$ is as follows:



Maps $m, M\colon \mathrm{Ch}(X) \to X$ is defined by the inner of a simplex $[x_0, \dots, x_n]$ being sent to x_0 respectively x_n by m respectively M. The C^*-algebras R_Y over X are then defined by $R_Y(Z) = C_0(m^{-1}(Y) \cap M^{-1}(Z))$ for all $Y, Z \in \mathbb{LC}(X)$. For the space $\mathcal{W}$, the fibres $m^{-1}(x)$ and $M^{-1}(x)$ for $x \in \mathcal{W}$ are the following



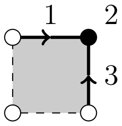
where a white dot denotes a point not belonging to the fibre.

For $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$, an extension $R_{Y \backslash U} \hookrightarrow R_Y \twoheadrightarrow R_U$ is obtained as $m^{-1}(U) \cap M^{-1}(Z)$ is a closed subset of $m^{-1}(Y) \cap M^{-1}(Z)$ for all $Z \in \mathbb{LC}(X)$. Recall that for a separable C^*-algebra A over X, this extension induces the six-term exact sequence in K-theory of the extension $A(U) \hookrightarrow A(Y) \twoheadrightarrow A(Y \backslash U)$.

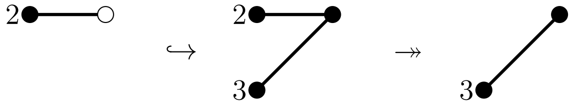
2.7.1. A concrete description of the representing object R_{12344}. In this section, a more concrete description of the new representing object R_{12344} for the filtered K-theory over the space $\mathcal{W}$ is given; cf. Section 2.5.2. The C^*-algebra R_{12344} is defined as the mapping cone of a generator of $\mathcal{NT}(234, 14)$ and at the first glance it is a bit surprising that one gets the same C^*-algebra if one chooses a generator of $\mathcal{NT}(134, 24)$ or of $\mathcal{NT}(124, 34)$ instead. In the following, it will be clearer why the choice between these three groups does not matter, and a sketch of proof will be given for the exactness of the six-term sequences used in the proof of Theorem 2.6.1.

The C^*-algebras over $\mathcal{W}$ that will be dealt with here are commutative and with a spectrum that can be embedded in $\mathbb{R}^2$, so they can be defined by drawing their spectrum. Ideals correspond to closed subsets of the spectrum, so the structure as a C^*-algebra over $\mathcal{W}$ is specified by marking which closed subsets of the spectrum that correspond to the open subsets of $\mathcal{W}$. Finally, embeddings of open subsets give injective $*$-homomorphisms and quotients to closed subsets give surjective $*$-homomorphisms.

As an example, the figure



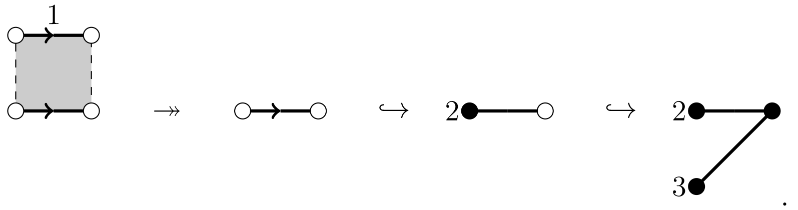
represents the C^*-algebra A over $\mathcal{W}$ defined by $A(4) = C_0((0,1) \times (0,1))$, $A(14) = C_0((0,1) \times (0,1])$, $A(24) = C_0((0,1) \times (0,1) \cup \{(1,1)\})$ and $A(34) = C_0((0,1] \times (0,1))$. The arrows indicate how the open interval $(0,1)$ is oriented, and the numbers indicate the structure as a C^*-algebra over $\mathcal{W}$. The orientation of the interval $(0,1)$ matters when one desires to calculate induced maps on K-theory. As another example, the extension $R_2 \hookrightarrow R_{234} \twoheadrightarrow R_{34}$ is drawn as follows:



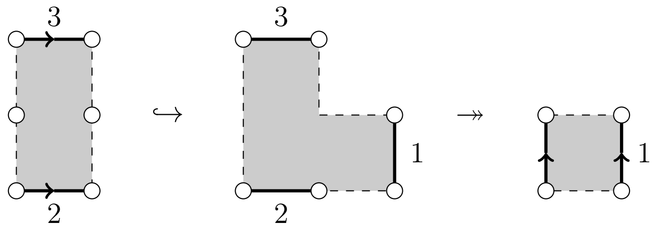
A generator of $\mathcal{NT}(234, 14) = \mathbb{Z}[1]$ is $r^2_{234}\delta^4_2 i^{14}_4$. The transformation i^{14}_4 is given by the restriction $R_{14} \twoheadrightarrow R_4$, and the transformation r^2_{234} is given by the embedding $R_2 \hookrightarrow R_{234}$ above. The transformation δ^4_2 is given by the extension $R_2 \hookrightarrow R_{24} \twoheadrightarrow R_4$ and is therefore at first more difficult to draw.

The transformation δ^4_2 is defined such that the extension $R_2 \hookrightarrow R_{24} \twoheadrightarrow R_4$ induces the element $\hat{\delta}^4_2$ in $KK_1(\mathcal{W}; R_4, R_2) = KK_0(\mathcal{W}; \mathrm{S}R_4, R_2)$. Since R_2 is projective, the map $KK_0(\mathcal{W}; \mathrm{S}R_4, R_2) \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(\mathrm{S}R_4), \mathrm{FK}(R_2))$ is an isomorphism, so if $\mathrm{FK}(\hat{\delta}^4_2) = \mathrm{FK}(\varphi)$ for some $\varphi\colon \mathrm{S}R_4 \to R_2$, we can conclude that $\bar{\varphi} = \delta^4_2$. One can calculate that $\mathrm{FK}(\hat{\delta}^4_2) = \pm\,\mathrm{FK}(\varphi)$ — depending on choice of Bott map — for $\varphi\colon \mathrm{S}R_4 \to R_2$ defined by

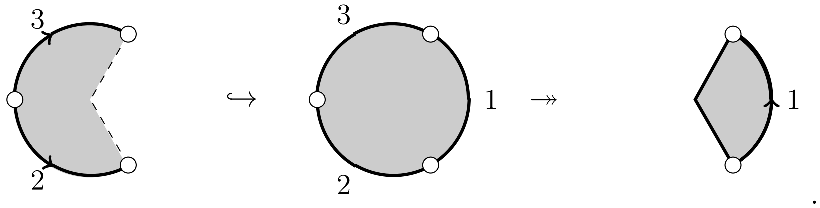
So $\mathcal{NT}(234, 14)$ is generated by $\alpha\colon \mathrm{S}R_{14} \to R_{234}$ given by



Up to $\mathcal{W}$-equivariant isomorphism of the mapping cone $\triangle_\alpha = \{(x, y) \in C_0((0,1], R_{234}) \oplus \mathrm{S}R_{14} \mid x(1) = \alpha(y)\}$, one can draw the mapping cone extension $\mathrm{S}R_{234} \hookrightarrow \triangle_\alpha \twoheadrightarrow \mathrm{S}R_{14}$ as

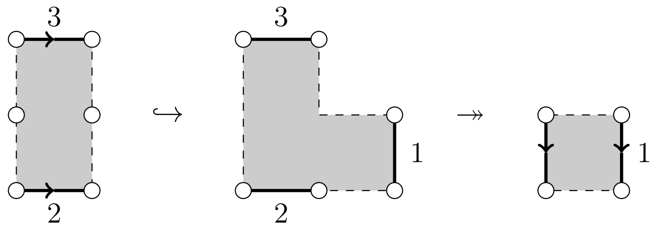


which up to $\mathcal{W}$-equivariant homotopy is

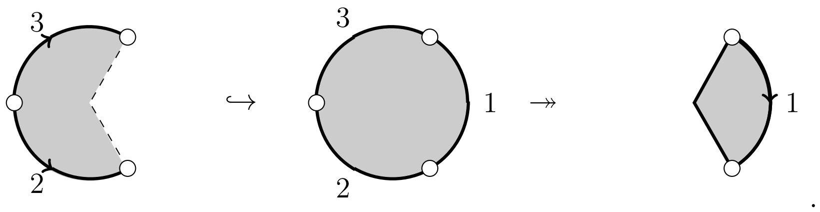


(2.1)

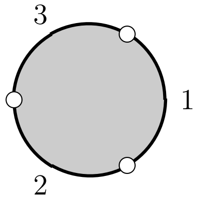
Had one instead of the generator $\bar{\alpha}$ of $\mathcal{NT}(234, 14)$ used the generator $-\bar{\alpha}$, one would have gotten the mapping cone extension



which up to $\mathcal{W}$-equivariant homotopy is

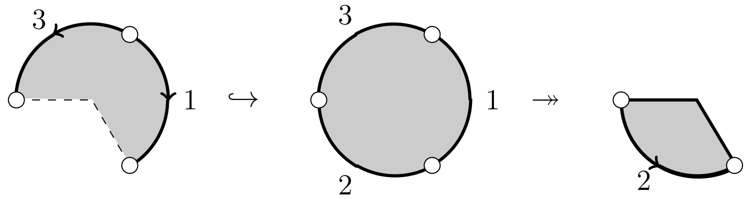


So by Meyer and Nest's definition of R_{12344}, R_{12344} is — up to $KK(\mathcal{W})$-equivalence — the suspension of the commutative C^*-algebra with spectrum



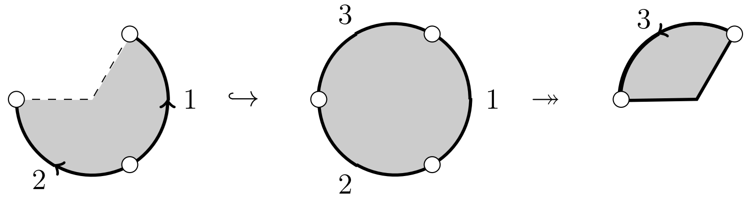
.

By repeating the construction using the generator $r^3_{134}\delta^4_3 i^{24}_4$ of $\mathcal{N}\mathcal{T}(134, 24)$ instead, one gets — up to $KK(\mathcal{W})$-equivalence — the mapping cone extension



. (2.2)

And by using the generator $r^1_{124}\delta^4_1 i^{34}_4$ of $\mathcal{N}\mathcal{T}(124, 34)$, one gets — up to $KK(\mathcal{W})$-equivalence — the mapping cone extension



. (2.3)

The six new $KK(\mathcal{W})$-classes arising from the three mapping cone extensions (2.1), (2.2) and (2.3) are generators for the six cyclic groups $\mathcal{N}\mathcal{T}(k4, 12344)$ and $\mathcal{N}\mathcal{T}(12344, ij4)$ — one sees this by applying $KK_*(\mathcal{W}; R_{12344}, -)$ and $KK_*(\mathcal{W}; -, R_{12344})$ to the three extensions.

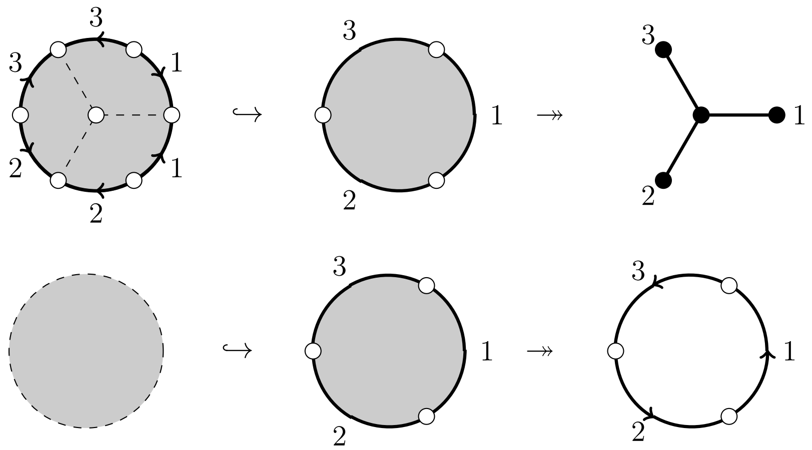
In the article [**ARR**] it is shown that there exist two exact triangles

$$\begin{tikzcd}
R_{1234} \arrow[dr, "(\hat{i}^{1234}_{ij4})"'] & & R_{12344} \arrow[ll, "\hat{r}^2_{1234}\hat{\delta}^4_2\hat{i}^{14}_4\hat{f}_{14}"', "\circ" marking] \\
& R_{124} \oplus R_{134} \oplus R_{234} \arrow[ur, "(\hat{f}^{ij4})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
R_{12344} \arrow[dr, "(\hat{f}_{k4})"'] & & R_4 \arrow[ll, "\hat{f}^{234}\hat{r}^2_{234}\hat{\delta}^4_2"', "\circ" marking] \\
& R_{14} \oplus R_{24} \oplus R_{34} \arrow[ur, "(\hat{i}^{k4}_4)"'] &
\end{tikzcd}$$

with each f^{ij4} generating the group $\mathcal{N}\mathcal{T}(12344, ij4)$ respectively, and with each f_{k4} generating $\mathcal{N}\mathcal{T}(k4, 12344)$ respectively. These triangles induce the six-term exact sequences used in the proof of Theorem 2.6.1.

In [**ARR**] this is proved by showing that the two triangles arise as mapping cone extensions — an abstract strategy that is easy to reuse for other spaces than the space $\mathcal{W}$. One can also show that the two extensions of commutative C^*-algebras



induce the desired exact triangles $\mathrm{S}R_{124} \oplus \mathrm{S}R_{134} \oplus \mathrm{S}R_{234} \to \mathrm{S}R_{12344} \to R_{1234} \to \mathrm{S}(\mathrm{S}R_{124} \oplus \mathrm{S}R_{134} \oplus \mathrm{S}R_{234})$ and $\mathrm{S}^2 R_4 \to \mathrm{S}R_{12344} \to \mathrm{S}R_{14} \oplus \mathrm{S}R_{24} \oplus \mathrm{S}R_{34} \to \mathrm{S}(\mathrm{S}^2 R_4)$. One sees that the correct $KK(\mathcal{W})$-classes are induced by keeping track of what happens to generators of the K-theory and by the same method as in the proof in [**ARR**]. However, this more concrete strategy is far less reusable.

CHAPTER 3

Classification of graph algebras

In this chapter, the notion of a graph algebra is defined, and an overview of the known results relevant for classification of graph algebras using filterered K-theory is given.

3.1. Graph algebras

A *countable, directed graph* $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices and a countable set E^1 of edges together with source and range maps $r, s\colon E^1 \to E^0$. If E^0 and E^1 are finite, we call E *finite*. We call E *row-finite* if $r^{-1}(v)$ is finite for all vertices $v \in E^0$, and a vertex $v \in E^0$ is called *regular* if $r^{-1}(v)$ is finite and nonempty. For a countable directed graph E, the relations

$$\begin{aligned} p_v &= p_v^* = p_v^2 \\ p_v p_w &= 0 \text{ when } v \neq w \\ s_e^* s_e &= p_{s(e)} \\ p_v &= \sum_{e \in r^{-1}(v)} s_e s_e^* \text{ when } 0 < |r^{-1}(v)| < \infty \end{aligned}$$

in $(p_v)_{v \in E_0}$ and $(s_e)_{s \in E^1}$ are bounded and closed, hence the universal C^*-algebra generated by these relations exists, and we denote it $C^*(E)$. A *graph algebra* is then a C^*-algebra of the form $C^*(E)$ for some countable, directed graph E.

In the litterature, two conflicting — but equivalent — definitions of $C^*(E)$ are used, depending on whether $p_{s(e)}$ or $p_{r(e)}$ is required to be the source projection of s_e. Here the convention used by I. Raeburn (cf. [**Rae05**]) is followed.

By construction, all graph algebras are separable. By [**Kat04**, 6.1, 6.6] all graph algebras are nuclear and lie in the bootstrap class of J. Rosenberg and R. Schochet. In the following, all graphs will be assumed to be countable and directed.

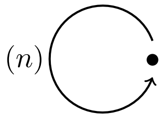
The *adjacency matrix* A_E of E is the $E^0 \times E^0$ matrix defined by

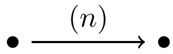
$$A_E(v, w) = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|.$$

Note that $E \mapsto A_E$ defines a one-to-one correspondence between countable directed graphs and, possibly infinite, square matrices over $\{0, 1, \ldots, \infty\}$ up to conjugacy with a permutation matrix.

When E is finite and pleasantly small, one can easily draw E. The graphs $E_1 = (E_1^0, E_1^1, r_1, s_1)$ and $E_2 = (E_2^0, E_2^1, r_2, s_2)$ defined by $E_1^0 = \{v\}$ and $E_1^1 = \{e_1, \ldots, e_n\}$ with $s(e_i) = r(e_i) = v$, and $E_2^0 = \{v_1, v_2\}$, and $E_2^1 =$

$\{e_1, \ldots, e_n\}$ with $s(e_i) = v_1$ and $r(e_i) = v_2$, are then drawn as follows.

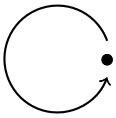


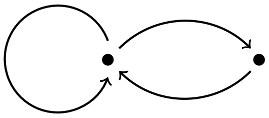


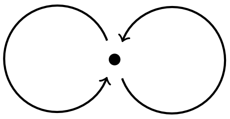
Their adjacency matrices are $\begin{pmatrix} n \end{pmatrix}$ and $\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$, respectively. One can show that $C^*(E_1)$ is isomorphic to the Cuntz algebra O_n and that $C^*(E_2)$ is isomorphic to the finite dimensional C^*-algebra M_{n-1}.

3.1.1. Condition (K). An important notion for graphs is condition (K). A *path* in E from v to w, for $v, w \in E^0$, is a finite sequence $e_1, \ldots, e_n$ of edges in E^1 satisfying $s(e_i) = r(e_{i+1})$ for all $i < n$, $r(e_1) = w$, and $s(e_n) = v$. A graph E is said to satisfy *condition (K)* if for all $v \in E^0$ either there is no *loop* based in v, i.e., there is no path from v to v, or there are two distinct *return paths* in v, i.e., there are distinct paths $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ from v to v with $r(e_i) \neq v$ for all $i < n$ and $r(f_i) \neq v$ for all $i < m$.

Consider the three examples below. The first graph does not satisfy property (K) — since there are many loops but only one return path based in its single vertex — while the second and the third do. Their associated C^*-algebras are $C(S^1)$, O_2, and O_2, respectively.





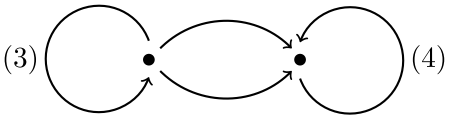


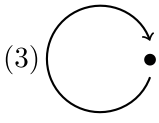
The simplest form of a return path is a cycle. A *cycle* based in $v \in E^0$ is an edge $e \in E^1$ satisfying $s(e) = v = r(e)$. In the examples above, the first two graphs contain one cycle each, while the third contains two.

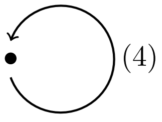
3.1.2. The ideal structure of a graph algebra. The ideal structure of $C^*(E)$ is reflected in the graph E. We define a preorder on E^0 by writing $w \leq v$ when there is a path from v to w. A subset H of E^0 is called *hereditary* if $H \ni w \leq v$ implies $v \in H$; and it is *saturated* if $r^{-1}(v) \neq \emptyset$ and $s(r^{-1}(v)) \subseteq H$ implies $v \in H$.

For a saturated, hereditary subset H of E^0, we consider the subgraphs $E_H = (H, r^{-1}(H), r, s)$ and $E \backslash H = (E^0 \backslash H, s^{-1}(E^0 \backslash H), r, s)$. If E satisfies condition (K), then so do E_H and $E \backslash H$.

EXAMPLE 3.1.1. Consider the graph E defined by $E^0 = \{v, w\}$ and $E^1 = \{e_1, e_2, e_3, e_4, f_1, f_2, g_1, g_2, g_3\}$ with $s(e_i) = r(e_i) = r(f_i) = v$ and $s(g_i) = r(g_i) = s(f_i) = w$. Then E has one nontrivial saturated hereditary subset, namely $H = \{w\}$. The three graphs E, E_H, and $E \backslash H$ are as follows.







By the theorem below, $C^*(E)/I_H$ is isomorphic to O_4, and I_H is stably isomorphic to O_3.

THEOREM 3.1.2 ([**Rae05**, 4.9]). *Let E be a row-finite, countable, directed graph satisfying condition (K). Then there is a lattice isomorphism between ideals in $C^*(E)$ and saturated hereditary subsets in E^0, given by mapping an ideal I to $H_I = \{v \in E^0 \mid p_v \in I\}$, and by mapping a saturated hereditary subsets H to the ideal I_H generated by $\{p_v \mid v \in H\}$. The quotient $C^*(E)/I_H$ is isomorphic to $C^*(E\backslash H)$, and $C^*(E_H)$ is isomorphic to a full corner in I_H.*

In particular, ideals and quotients of graph algebras are graph algebras up to stable isomorphism, hence subquotients of graph algebras are graph algebras up to stable isomorphism. Also, if E^0 is finite, then $C^*(E)$ has finitely many ideals. So the simple subquotients of a graph algebra are simple graph algebras up to stable isomorphism. And by [**KPR98**, 3.11], a simple graph algebra is either an AF algebra or a Kirchberg algebra. Notice that all simple graph algebras have real rank zero.

Recall that for an AF algebra A, $(K_0(A), K_0(A)^+)$ is a dimension group, while for a Kirchberg algebra A, $K_0(A)$ equals $K_0(A)^+$. So by the classification results of G. A. Elliott, and N. C. Phillips and E. Kirchberg, simple graph algebras are classified by ordered K-theory $(K_*(-), K_0(-)^+)$ as the positive cone lets us determine if it is an AF algebra or a Kirchberg algebra.

3.1.3. Real rank zero, pure infiniteness, and K-theory.

It is also reflected in the graph whether the graph algebra has real rank zero and whether it is purely infinite.

A vertex $v \in E^0$ is called a *breaking vertex* if $|r^{-1}(v)| = \infty$ while the set $r^{-1}(v)\backslash s^{-1}(\{w \neq v \mid w \not\leq v\})$ is finite and nonempty. If the graph E is row-finite, then there are no breaking vertices in E. A subset $M \subseteq E^0$ is called a *maximal tail* in E if the following three conditions are satisfies:

1. If $w \in M$ and $v \leq w$, then $v \in M$.
2. If $v \in M$ and $0 < |r^{-1}(v)| < \infty$, then there exists an edge $e \in E^1$ satisfying $r(e) = v$ and $s(e) \in M$.
3. For all $v, w \in M$, there exists $y \in M$ satisfying $v \leq y$ and $w \leq y$.

THEOREM 3.1.3 ([**HS03**, 2.3, 2.5]). *Let E be a countable, directed graph. Then $C^*(E)$ has real rank zero if and only if E satisfies condition (K).*

And then $C^*(E)$ is purely infinite if and only if E satisfies condition (K), E has no breaking vertices, and for each vertex v in each maximal tail M in E there is a path to v from a return path in E.

Notice that for graph algebras, pure infiniteness implies real rank zero. Notice also that if all vertices in E are regular and support at least two return paths, then $C^*(E)$ is purely infinite.

The *Cuntz-Krieger algebras* are the graph algebras arising from finite graphs with adjacency matrices over $\{0, 1\}$. Equivalently, the Cuntz-Krieger algebras are the graph algebras arising from finite graphs E with the property that $s^{-1}(v) \neq \emptyset$ and $r^{-1}(v) \neq \emptyset$ holds for all $v \in E^0$; cf. [**EW80**]. In particular, a graph algebra of a finite graph with at least one cycle based in each vertex is a Cuntz-Krieger algebra. The Cuntz-Krieger algebras satisfying condition (K) are purely infinite.

The K-theory of a graph algebra is also reflected in the underlying graph.

THEOREM 3.1.4 ([**Rae05**, 7.16]). *Let E be a countable, directed graph and assume that all vertices in E are regular. Let A_E be the adjacency matrix of E. Then $K_0(C^*(E))$ and $K_1(C^*(E))$ are isomorphic to the cokernel and kernel, respectively, of the map $\mathbb{Z}^{E^0} \overset{A_E-1}{\longrightarrow} \mathbb{Z}^{E^0}, x \mapsto xA_E - x$.*

A formula for arbitrary graphs is given in, e.g., [**RS04**]. In particular, the K_1-group of a graph algebra is always free. By the following theorem, also the positive cone $K_0^+(C^*(E))$ in $K_0(C^*(E))$ is reflected in the underlying graph.

THEOREM 3.1.5 ([**AMP07**, 7],[**Tom03**]). *Let E be a countable, directed graph and assume that all vertices in E are regular. Let A_E be the adjacency matrix of E. Then the isomorphism $K_0(C^*(E)) \to \operatorname{coker}(A_E - 1)$ maps the positive cone $K_0^+(C^*(E))$ onto the subset $\mathbb{Z}_+^{E^0} / \operatorname{im}(A_E - 1)$.*

A graph E is called *transitive* if $v \leq w$ and $w \leq v$ holds for all $v, w \in E^0$. By a theorem of W. Szymański (cf. [**Szy02**]) any pair (G, F) of countable, abelian groups with F free, can be realized as the K-theory of a graph algebra $C^*(E)$ with E transitive and row-finite. By a result in [**EKTW**], the graph E can be chosen such that furthermore every vertex is the base of at least two cycles, and E can be chosen to be finite given that G and F are finitely generated with $\operatorname{rank} G = \operatorname{rank} F$. Hence the pair (G, F) can be realized as the K-theory of a simple, purely infinite graph algebra, and if G and F are finitely generated with $\operatorname{rank} G = \operatorname{rank} F$, then even as the K-theory of a simple Cuntz-Krieger algebra of real rank zero.

3.2. Classification of graph algebras using filtered K-theory

For classification of nonsimple graph algebras, filtered K-theory and reductions thereof seem suitable.

Let X be a finite T_0-space. As X is finite, there exists for each subset Y of X a smallest open subset of X containing Y; we refer to this as the *opener* of Y and denote it $\widetilde{Y}$. The *open boundary* $\widetilde{\partial}(Y)$ of Y is defined as $\widetilde{Y} \backslash Y$. Recall from Section 2.4 that we write $x \to y$ for $x, y \in X$ when x is a closed point in $\widetilde{\partial}(y)$.

In [**ABK**], an invariant $\mathrm{FK}_{\mathcal{R}}$ is defined for C^*-algebras A over X to consist of the groups and maps

$$K_1(A(x)) \xrightarrow{\delta} K_0(A(\widetilde{\partial}(x))) \xrightarrow{i} K_0(A(\widetilde{\{x\}}))$$

for all $x \in X$, together with the groups and maps

$$K_0(A(\widetilde{\{y\}})) \xrightarrow{i} K_0(A(\widetilde{\partial}(x)))$$

for all $x, y \in X$ for which $y \to x$. For tight C^*-algebras over X, this definition coincides with the definition of reduced filtered K-theory by G. Restorff in [**Res06**], except for two things. First, there is a redundancy in $\mathrm{FK}_{\mathcal{R}}$ when $\widetilde{\{y\}}$ equals $\widetilde{\partial}(x)$ and the map $K_0(A(\widetilde{\{y\}})) \xrightarrow{i} K_0(A(\widetilde{\partial}(x)))$ becomes an isomorphism. Second, G. Restorff includes the group $K_0(A(x))$ and the map $K_0(A(\widetilde{\{x\}})) \xrightarrow{r} K_0(A(x))$ for all $x \in X$, but for a real rank zero C^*-algebra A these are naturally isomorphic to the cokernel of $K_0(A(\widetilde{\partial}(x))) \xrightarrow{i} K_0(A(\widetilde{\{x\}}))$. Hence the invariants $\mathrm{FK}_{\mathcal{R}}$ and reduced filtered K-theory are not equal but for tight C^*-algebras of real rank zero they are equivalent, hence $\mathrm{FK}_{\mathcal{R}}$ is also referred to as *reduced filtered K-theory*.

A finite T_0-space X has the *unique path property* if for any $x, y \in X$, $x \to x_1 \to \cdots \to x_n \to y$ and $x \to x'_1 \to \cdots \to x'_{n'} \to y$ implies $n = n'$ and $x_i = x'_i$ for all i. All accordion spaces (cf. Theorem 2.4.1) have the unique path property, and so do the spaces $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{S}$ considered in Section 2.5. The space $\mathcal{D}$ considered in Section 2.5 does not have the unique path property.

For a space X with the unique path property, an invariant $\mathrm{FK}_{\mathcal{B}}$ for C^*-algebras A over X was defined in [**ABK**] to consist of the groups

$$K_1(A(\overline{\{x\}})), K_0(A(\widetilde{\{x\}}))$$

for all $x \in X$, together with the maps

$$K_1(A(\overline{\{x\}})) \xrightarrow{r} K_1(A(\overline{\{y\}})) \xrightarrow{\delta} K_0(A(\widetilde{\{x\}})) \xrightarrow{i} K_0(A(\widetilde{\{y\}}))$$

for all $x \to y$. The specified maps in $\mathcal{NT}$ exist since X has the unique path property. The invariant $\mathrm{FK}_{\mathcal{B}}$ is referred to as *filtered K-theory restricted to the canonical base*.

The invariants $\mathrm{FK}_{\mathcal{R}}$ and $\mathrm{FK}_{\mathcal{B}}$ are strictly coarser than FK.

Example 3.2.1. For the spaces $\mathcal{W}$ and $\mathcal{D}$ we compare $\mathrm{FK}_{\mathcal{R}}$ and $\mathrm{FK}_{\mathcal{B}}$ with FK. We restrict to real rank zero C^*-algebras, where boundary maps from even to odd parts vanish, to make the diagrams simpler and since $\mathrm{FK}_{\mathcal{B}}$ and $\mathrm{FK}_{\mathcal{R}}$ are only useful invariants for real rank zero C^*-algebras. For the space $\mathcal{W}$, the invariant $\mathrm{FK}_{\mathcal{R}}$ consists of the groups and maps

$$\begin{tikzcd}
 & 3_1 \arrow[dr] & & 14_0 \\
4_1 & 2_1 \arrow[r] & 4_0 \arrow[ur] \arrow[r] \arrow[dr] & 24_0 \\
 & 1_1 \arrow[ur] & & 34_0
\end{tikzcd},$$

while $\mathrm{FK}_{\mathcal{B}}$ consists of the groups and maps

$$\begin{tikzcd}
 & 3_1 \arrow[dr] & & 14_0 \\
1234_1 \arrow[ur] \arrow[r] \arrow[dr] & 2_1 \arrow[r] & 4_0 \arrow[ur] \arrow[r] \arrow[dr] & 24_0 \\
 & 1_1 \arrow[ur] & & 34_0
\end{tikzcd},$$

and FK for real rank zero C^*-algebras consists of the groups and maps

$$\begin{tikzcd}
 & 34_1 \arrow[r] \arrow[dr] & 234_1 \arrow[dr] & & 3_1 \arrow[dr] & & 14_0 \arrow[r] \arrow[dr] & 124_0 \arrow[dr] & & 1_0 \\
4_1 \arrow[ur] \arrow[r] \arrow[dr] & 24_1 \arrow[ur] \arrow[dr] & 134_1 \arrow[r] & 1234_1 \arrow[ur] \arrow[r] \arrow[dr] & 2_1 \arrow[r] & 4_0 \arrow[ur] \arrow[r] \arrow[dr] & 24_0 \arrow[ur] \arrow[dr] & 134_0 \arrow[r] & 1234_0 \arrow[ur] \arrow[r] \arrow[dr] & 2_0 \\
 & 14_1 \arrow[r] \arrow[ur] & 124_1 \arrow[ur] & & 1_1 \arrow[ur] & & 34_0 \arrow[r] \arrow[ur] & 234_0 \arrow[ur] & & 3_0
\end{tikzcd}.$$

And for the space $\mathcal{D}$, the invariant $\mathrm{FK}_{\mathcal{R}}$ consists of the groups and maps

$$\begin{tikzcd}
 & 3_1 \arrow[dr] & & 24_0 \arrow[dr] & & \\
4_1 & & 4_0 \arrow[ur] \arrow[dr] & 1_1 \arrow[r] & 234_0 \arrow[r] & 1234_0 \\
 & 2_1 \arrow[ur] & & 34_0 \arrow[ur] & &
\end{tikzcd},$$

while FK for real rank zero C^*-algebras consists of the groups and maps

$$\begin{tikzcd}
 & 34_1 \arrow[dr] & & 3_1 \arrow[dr] & & 13_1 \arrow[r] \arrow[dr] & 24_0 \arrow[dr] & & 2_0 \arrow[dr] & & 12_0 \arrow[dr] & \\
4_1 \arrow[ur] \arrow[dr] & & 234_1 \arrow[r] \arrow[ur] \arrow[dr] & 1234_1 \arrow[r] & 123_1 \arrow[r] \arrow[ur] \arrow[dr] & 4_0 \arrow[ur] \arrow[dr] & 1_1 \arrow[r] & 234_0 \arrow[r] \arrow[ur] \arrow[dr] & 1234_0 \arrow[r] & 123_0 \arrow[ur] \arrow[dr] & & 1_0 \\
 & 24_1 \arrow[ur] & & 2_1 \arrow[ur] & & 12_1 \arrow[r] \arrow[ur] & 34_0 \arrow[ur] & & 3_0 \arrow[ur] & & 13_0 \arrow[ur] & .
\end{tikzcd}$$

The invariant $\mathrm{FK}_{\mathcal{B}}$ is not defined for C^*-algebras over $\mathcal{D}$ as $\mathcal{D}$ does not have the unique path property. The groups $\widetilde{\{x\}}_0$ and $\overline{\{x\}}_1$, for $x \in \mathcal{D}$, are 4_0, 34_0, 24_0, 1234_0, 1234_1, 13_1, 12_1, and 1_1. Notice that the maps $\delta_{13}^4, \delta_{12}^3, \delta_1^{34}, \delta_1^{24}$ do not exist in $\mathcal{NT}$ as 134 and 124 are not locally closed subsets of $\mathcal{D}$.

3.2.1. Classification of Cuntz-Krieger algebras. G. Restorff defined the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ in order to classify Cuntz-Krieger algebras, and proved the following classification result, reformulated in our terms.

THEOREM 3.2.2 ([**Res06**, 4.2]). *Let X be any finite T_0-space, and let A and B be Cuntz-Krieger algebras that are tight over X. If* $\mathrm{FK}_{\mathcal{R}}(A)$ *and* $\mathrm{FK}_{\mathcal{R}}(B)$ *are isomorphic, then $A \otimes \mathbb{K}$ and $B \otimes \mathbb{K}$ are isomorphic.*

Two things should be noted about this result. First, this is not a strong classification, i.e., it does not allow us to lift isomorphisms on $\mathrm{FK}_{\mathcal{R}}$ to stable isomorphisms. Second, the proof is based on work by M. Boyle and D. Huang on shift spaces, [**Boy02**] and [**BH03**], using that any Cuntz-Krieger algebra has an underlying shift space; so the result does not allow us to compare Cuntz-Krieger algebras with more general C^*-algebras with the same $\mathrm{FK}_{\mathcal{R}}$, and the proof cannot be generalized beyond the class of Cuntz-Krieger algebras. So the result of G. Restorff does not tell us whether phantom Cuntz-Krieger algebras exist over general X. A *phantom Cuntz-Krieger algebra* over X is a tight, purely infinite, nuclear C^*-algebra A in $\mathcal{B}(X)$ which is not stably isomorphic to a Cuntz-Krieger algebra and yet satisfies the property that $\mathrm{FK}(A)$ is isomorphic to $\mathrm{FK}(B)$ for some Cuntz-Krieger algebra B which is tight over X.

Under some restrictions on X, the result of G. Restorff can be improved slightly in the following way. As Cuntz-Krieger algebras are separable, nuclear, and purely infinite and satisfy the property that all simple subquotients lie in the bootstrap class, the results of E. Kirchberg, R. Meyer and R. Nest, and R. Bentmann apply to Cuntz-Krieger algebras; cf. Chapter 2. Hence for Cuntz-Krieger algebras with primitive ideal space isomorphic to an accordion space X, isomorphisms on FK lift to stable isomorphisms, and we can compare these Cuntz-Krieger algebras with tight, purely infinite, nuclear C^*-algebras in $\mathcal{B}(X)$. In Section 3.4, the significance of the fact that there are no phantom Cuntz-Krieger algebras over X will be clarified.

3.2.2. Conjecture for graph algebras. The work of R. Meyer and R. Nest in [**MN**] together with the classification result for Cuntz-Krieger algebras of G. Restorff in [**Res06**], have inspired S. Eilers, G. Restorff, and E. Ruiz to conjecture in [**ERR**] that ordered filtered K-theory FK^+ classifies real rank zero graph algebras with finitely many ideals. *Ordered filtered K-theory* $\mathrm{FK}^+(A)$ for a C^*-algebra A over X, consists of $\mathrm{FK}_{\mathcal{ST}}(A)$ together with the positive cones $K_0^+(A(Y))$ for all $Y \in \mathbb{LC}(X)$. An isomorphism on ordered filtered K-theory is then an $\mathcal{ST}$-isomorphism that restricts to order isomorphisms on the even parts of the groups. As ordered K-theory classifies the simple graph algebras, and as the simple subquotients of a graph algebra are again graph algebras, the invariant $\mathrm{FK}^+(A)$ tells us what the simple subquotients of A are. The intuitive idea is then that $\mathrm{FK}^+(A)$ contains enough information to tell us how these simple subquotients are glued together to form A.

So far, there are no counterexamples to the conjecture of S. Eilers, G. Restorff, and E. Ruiz, and in [**ERR**] they establish the following partial result.

THEOREM 3.2.3 ([**ERR**, 6.9]). *Let $X = \{x_1, \dots, x_n\}$ be a finite linear T_0-space with $x_j \leq x_i$ when $j \geq i$. Let A and B be tight graph algebras over X of real rank zero, and assume that there exists an i such that either $A(x_1), \dots, A(x_i)$ are purely infinite and $A(x_{i+1}), \dots, A(x_n)$ are AF algebras, or $A(x_1), \dots, A(x_i)$ are AF algebras and $A(x_{i+1}), \dots, A(x_n)$ are purely infinite.*

If $\mathrm{FK}^+(A)$ and $\mathrm{FK}^+(B)$ are isomorphic, then $A \otimes \mathbb{K}$ and $B \otimes \mathbb{K}$ are isomorphic.

Their proof combines the UCT of R. Meyer and R. Nest, Theorems 2.3.6 and 2.4.1, with a modification of the proof of E. Kirchberg for Theorem 2.1.1. Given a $KK(X)$-equivalence between tight graph algebras over a space X, they can construct a stable isomorphism between the graph algebras, provided the $KK(X)$-equivalence induces positive maps on the K_0-groups, and provided the ideal lattice satisfies some technical conditions which in the linear case holds under the conditions stated above. Note that S. Eilers, G. Restorff, and E. Ruiz are not able to lift the $KK(X)$-equivalence but are only able to use the existence of a $KK(X)$-equivalence to construct some stable isomorphism.

3.2.3. Partial classification results. According to E. Ruiz, the modification by S. Eilers, G. Restorff, and E. Ruiz of the proof of E. Kirchberg also works for the spaces $\mathcal{W}_n = \{x_0, x_1, \dots, x_n\}$ with $\mathcal{O}(\mathcal{W}_n) = \{U \subseteq \mathcal{W}_n \mid x_0 \in U\} \cup \{\emptyset\}$, with no restriction on the position of simple, purely infinite subquotients in the ideal lattice, by [**Rui10**]. To complete the proof for the spaces $\mathcal{W}_n$, S. Eilers, G. Restorff, and E. Ruiz would need to be able to lift isomorphisms on FK to $KK(\mathcal{W}_n)$-equivalences, at least for C^*-algebras with the same filtered K-theory as graph algebras. In [**ARR**] this is done for real rank zero C^*-algebras over $\mathcal{W}_3 = \mathcal{W}$; cf. Theorem 2.6.1.

The only known real rank zero counterexample is for the space $\mathcal{D}$, and in [**ABK**] the following is proved. As a consequence, there are no phantom Cuntz-Krieger algebras over $\mathcal{D}$, and all tight, purely infinite graph algebras over $\mathcal{D}$ are classified by reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$.

THEOREM 3.2.4 ([**ABK**, 8.15]). *Let A and B be real rank zero C^*-algebras in the bootstrap class $\mathcal{B}(\mathcal{D})$ and assume that $K_1(A(x))$ and $K_1(B(x))$ are free for all $x \in \mathcal{D}$. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ lifts to a $KK(X)$-equivalence.*

Sketch of proof. By Theorem 2.5.4, it suffices to extend an isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ to an isomorphism $\varphi'\colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. Note that φ should be extended to the remaining groups in a way that respects the natural transformations.

In Example 3.2.1, it is recalled which groups and maps $\mathrm{FK}_{\mathcal{R}}(A)$ and $\mathrm{FK}_{\mathcal{R}}(B)$ consist of. The category $\mathcal{NT}'$ is

$$\begin{tikzcd}
 & 12 \arrow[dr, "f", "\circ" marking] & & 34 \arrow[dr, "i"] & & 3 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "\delta", "\circ" marking] \arrow[dr, "r"'] & 4 \arrow[r, "f"] & 4\backslash 1 \arrow[ur, "f"] \arrow[r, "f", "\circ" marking] \arrow[dr, "f"'] & 1 \arrow[r, "\delta", "\circ" marking] & 234 \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "r"'] & 1234 \arrow[r, "r"] & 123 \\
 & 13 \arrow[ur, "f"', "\circ" marking] & & 24 \arrow[ur, "i"'] & & 2 \arrow[ur, "i"'] &
\end{tikzcd}.$$

As in the proof of Theorem 2.6.1, the morphism φ can be extended to 2_0, 3_0, 1234_0, 123_0, 12_0, 13_0, and then 1_0 as the maps induced on cokernels. E.g., 2_0 is isomorphic to the cokernel of $34_0 \to 234_0$, and 123_0 is isomorphic to the cokernel of $234_0 \to 2_0 \oplus 1234_0 \oplus 3_0$, due to real rank zero.

The groups $4\backslash 1_0$, 13_1, 12_1, 123_1, 1234_1, 234_1, 34_1, and 24_1 can all be recovered as direct sums of groups and cokernels of maps appearing in $\mathrm{FK}_{\mathcal{R}}(A)$. E.g., 13_1 is isomorphic to $3_1 \oplus \ker(\delta_3^1\colon 1_1 \to 3_0)$, and 1234_1 is isomorphic to $4_1 \oplus 2_1 \oplus 3_1 \oplus \ker((\delta_1^2, \delta_1^3)\colon 1_1 \to 2_0 \oplus 3_0)$. The split maps must be chosen such that the natural transformations are preservered, and this is done by following the order specified, starting with $4\backslash 1_0$, and making sure that all natural transformations out of the group in question are respected. Finally for the group $4\backslash 1_1$, we note that it is (isomorphic to) the kernel of $34_1 \oplus 1_0 \oplus 24_1 \to 234_1$; cf. the proof of Theorem 2.6.1. ☐

In [**ABK**] it is noted that by construction, φ' is an order isomorphism on the groups $K_0(A(Y)) \to K_0(B(Y))$ for all $Y \in \mathbb{LC}(\mathcal{D})$ given that φ is for the groups 4_0, 24_0, 34_0, 234_0, and 1234_0.

The strategy of first lifting an isomorphism on FK to a $KK(X)$-equivalence using a modification of the results of R. Meyer and R. Nest, and then using the $KK(X)$-equivalence to construct a $*$-isomorphism using a modification of the result of E. Kirchberg, does not appear to be a useful strategy for general spaces X. The reason being that both steps depend on X and have to be dealt with one space at a time. Other methods are therefore needed for the general case. However, at this point it is extremely useful to establish partial results and provide examples in order to get a better understanding of the situation.

3.3. Calculating filtered K-theory for graph algebras

For graph algebras, there is a formula for calculating the filtered K-theory, by T. M. Carlsen, S. Eilers, and M. Tomforde in [**CET**], which generalizes the formula for Cuntz-Krieger algebras by G. Restorff in [**Res06**]. The formula for a general graph algebra is slightly complicated to write up, so here only the case of a graph E satisfying condition (K) and with all vertices regular is considered.

For such a graph E and a saturated hereditary subset H of E^0, we consider the adjacency matrix A_E for E, which is a $E^0 \times E^0$ matrix defined by

$$A_E(v,w) = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|$$

and the block parts of the matrix consisting of the $H \times H$ matrix A_H defined by $A_H(v,w) = A_E(v,w)$, the $E^0\backslash H \times E^0\backslash H$ matrix $A_{E\backslash H}$ defined by $A_{E\backslash H}(v,w) = A_E(v,w)$, and the $E^0\backslash H \times H$ matrix Y_H defined by $Y_H(v,w) = A_E(v,w)$. Notice that $A_E(v,w) = 0$ when $v \in H$ and $w \in E^0\backslash H$. We therefore have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^H \arrow[r] \arrow[d, "A_H"] & \mathbb{Z}^{E^0} \arrow[r] \arrow[d, "A_E"] & \mathbb{Z}^{E^0\backslash H} \arrow[r] \arrow[d, "A_{E\backslash H}"] & 0 \\
0 \arrow[r] & \mathbb{Z}^H \arrow[r] & \mathbb{Z}^{E^0} \arrow[r] & \mathbb{Z}^{E^0\backslash H} \arrow[r] & 0
\end{tikzcd}$$

with short exact rows, and by the Snake Lemma, this induces a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(A_H - 1) \arrow[r] & \ker(A_E - 1) \arrow[r] & \ker(A_{E\backslash H} - 1) \arrow[dll, rounded corners] & \\
& \operatorname{coker}(A_H - 1) \arrow[r] & \operatorname{coker}(A_E - 1) \arrow[r] & \operatorname{coker}(A_{E\backslash H} - 1) \arrow[r] & 0
\end{tikzcd}$$

where one can check that the map $\ker(A_{E^0\backslash H} - 1) \to \operatorname{coker}(A_H - 1)$ is induced by Y_H.

Recall Theorems 3.1.2, 3.1.4, and 3.1.5.

Theorem 3.3.1 ([**CET**, 4.1]). *Let E be a countable, directed graph satisfying condition (K) where all vertices are regular, and let H be a saturated hereditary subset of E^0.*

Then the six-term exact sequence in K-theory induced by the extension $I_H \hookrightarrow C^*(E) \twoheadrightarrow C^*(E)/I_H$ is naturally isomorphic to the sequence

$$\begin{tikzcd}
\operatorname{coker}(A_H - 1) \arrow[r] & \operatorname{coker}(A_E - 1) \arrow[r] & \operatorname{coker}(A_{E^0\backslash H} - 1) \arrow[d, "0"] \\
\ker(A_{E^0\backslash H} - 1) \arrow[u, "Y_H"] & \ker(A_E - 1) \arrow[l] & \ker(A_H - 1). \arrow[l]
\end{tikzcd}$$

Example 3.3.2. For the graph E with a unique nontrivial saturated hereditary subset H, considered in Example 3.1.1, the adjacency matrix is $\begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}$, and the six-term exact sequence in K-theory induced by the extension $I_H \hookrightarrow C^*(E) \twoheadrightarrow C^*(E)/I_H$ collapses to $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/3 \twoheadrightarrow \mathbb{Z}/3$ as all the K_1-groups vanish.

Example 3.3.3. Consider the graph E with adjacency matrix

$$A_E = \begin{pmatrix} 5 & 4 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 6 & 2 & 4 & 3 \\ 1 & 0 & 3 & 4 \end{pmatrix}.$$

Denote the unique nontrivial saturated hereditary subset of E^0 by H. Then the six-term exact sequence of $I_H \hookrightarrow C^*(E) \twoheadrightarrow C^*(E)/I_H$ is

$$\begin{tikzcd}
\mathbb{Z}/4 \oplus \mathbb{Z} \arrow[r, "\begin{pmatrix}1 & 6 & 0\\ 0 & 15 & 0\end{pmatrix}"] & \mathbb{Z}/2 \oplus \mathbb{Z}/24 \oplus \mathbb{Z} \arrow[r, "\begin{pmatrix}0 & 1\\ 2 & 2\\ 0 & 0\end{pmatrix}"] & \mathbb{Z}/3 \oplus \mathbb{Z} \arrow[d, "\begin{pmatrix}0\\ 0\end{pmatrix}"] \\
\mathbb{Z} \arrow[u, "\begin{pmatrix}2 & 4\end{pmatrix}"] & \mathbb{Z} \arrow[l, "\begin{pmatrix}0\end{pmatrix}"] & \mathbb{Z}. \arrow[l, "\begin{pmatrix}1\end{pmatrix}"]
\end{tikzcd}$$

The computation goes as follows. The three matrices

$$A_E - 1 = \begin{pmatrix}4 & 4 & 0 & 0\\ 4 & 4 & 0 & 0\\ 6 & 2 & 3 & 3\\ 1 & 0 & 3 & 3\end{pmatrix}, A_H - 1 = \begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}, A_{E\setminus H} - 1 = \begin{pmatrix}3 & 3\\ 3 & 3\end{pmatrix}$$

have Smith normal form

$$S_E = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 24 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, S_H = \begin{pmatrix}4 & 0\\ 0 & 0\end{pmatrix}, S_{E\setminus H} = \begin{pmatrix}3 & 0\\ 0 & 0\end{pmatrix},$$

respectively, from which their kernels and cokernels easily can be read. Given the invertible matrices U_E, V_E, U_H, V_H, $U_{E\setminus H}$, and $V_{E\setminus H}$ over $\mathbb{Z}$ for which

$$S_E = V_E(A_E - 1)U_E, S_H = V_H(A_H - 1)U_H, S_{E\setminus H} = V_{E\setminus H}(A_{E\setminus H} - 1)U_{E\setminus H},$$

one can calculate the induced maps between the kernels and cokernels, e.g., the induced map $\ker B_H \to \ker B_E$ as

$$V_H \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} V_E^{-1}$$

and the induced map $\ker B_{E\setminus H} \to \operatorname{coker} B_H$ as

$$V_{E\setminus H} \begin{pmatrix}6 & 2\\ 1 & 0\end{pmatrix} U_H.$$

Using the formula of Theorem 3.3.1, it is therefore possible to calculate the filtered K-theory of a graph algebra. However, even for small primitive ideal spaces, there is no known algorithm that determines whether two C^*-algebras have isomorphic filtered K-theory. One can only try to either construct an isomorphism, or to determine various algebraic properties that the two filtered K-theories do not share. In [**ABK**], it is shown that for suitably nice C^*-algebras, including the graph algebras of real rank zero, fewer groups and maps need to be calculated and compared. For this, the notion of boundary decomposition property is introduced in [**ABK**]. A space X with the unique path property is said to have the *boundary decomposition property* if for all $Y \in \mathbb{LC}(X)$ and all $U \in \mathcal{O}(Y)$,

$$\delta_C^U = \sum_{U \ni x \to y \in C} r_C^{C\cap\overline{\{y\}}} i_{C\cap\overline{\{y\}}}^{\overline{\{y\}}} \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} r_{\widetilde{\{x\}}}^{U\cap\widetilde{\{x\}}} i_{U\cap\widetilde{\{x\}}}^{U}$$

holds when $C = Y\backslash U$. This rather technical property guarantees that all boundary maps in $\mathcal{NT}$ are determined by restriction and extension maps together with the boundary maps that appear in $\mathrm{FK}_{\mathcal{B}}$.

The accordion spaces and the spaces $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{S}$ of Section 2.5 have the boundary decomposition property. Also, recall that for these spaces, the filtered K-theory FK and the concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$ coincide.

THEOREM 3.3.4 ([**ABK**, 6.10]). *Assume that X has the boundary decomposition property and let A and B be real rank zero C^*-algebras over X. Then any isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{B}}(A) \to \mathrm{FK}_{\mathcal{B}}(B)$ extends uniquely to an isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$. If φ is an order isomorphism, then so is Φ.*

SKETCH OF PROOF. As before, the extension to the remaining groups must respect the natural transformations. As in the proof of Theorem 2.6.1, this is done by extending to cokernels and kernels. By extending to cokernels, the claim on positivity is automatically satisfied.

For each open subset U of X,

$$\bigoplus_{z>x\in U} \mathrm{FK}^0_{\widetilde{\{z\}}}(A) \longrightarrow \bigoplus_{x\in U} \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \longrightarrow \mathrm{FK}^0_U(A) \longrightarrow 0$$

is exact as A has real rank zero and X has the unique path property, hence a map $\mathrm{FK}^0_U(A) \to \mathrm{FK}^0_U(B)$ is induced. For general $Y \in \mathbb{LC}(X)$, take open sets U and V such that $Y = U\backslash V$; then

$$\mathrm{FK}^0_V(A) \longrightarrow \mathrm{FK}^0_U(A) \longrightarrow \mathrm{FK}^0_Y(A) \longrightarrow 0$$

is exact and a map $\mathrm{FK}^0_Y(A) \to \mathrm{FK}^0_Y(B)$ is induced. For $\mathrm{FK}^1_Y(A) \to \mathrm{FK}^1_Y(B)$ a dual version for $(\overline{\{x\}})_{x\in X}$ and closed subsets applies. $\square$

THEOREM 3.3.5 ([**ABK**, 8.14]). *Assume that X has the boundary decomposition property and let A and B be real rank zero C^*-algebras over X. Assume that $K_1(A(x))$ and $K_1(B(x))$ are free groups for all $x \in X$. Then any isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ extends (nonuniquely) to an isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$. If φ is positive, then so is Φ.*

SKETCH OF PROOF. By Theorem 3.3.4 and its proof, it suffices to extend to $\mathrm{FK}^1_{\overline{\{x\}}}(A) \to \mathrm{FK}^1_{\overline{\{x\}}}(B)$ for all $x \in X$. This is done inductively over the ordering $\leq$ on X, using the same idea as in the proof of Theorem 3.2.4, starting with the closed points, and using that $\mathrm{FK}^1_{\overline{\{x\}}}(A)$ is isomorphic to the direct sum of $\mathrm{FK}^1_{\{x\}}(A)$ and a free subgroup of $\bigoplus_{y\to x} \mathrm{FK}^1_{\overline{\{y\}}}(A)$. Again, the split maps are chosen such that natural transformations out of $\overline{\{x\}}_1$ are respected. $\square$

EXAMPLE 3.3.6. Consider the graph E_1 with adjacency matrix

$$A_{E_1} = \begin{pmatrix} 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 5 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 5 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 4 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 4 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 & 0 & 7 & 6 \\ 1 & 1 & 0 & 0 & 0 & 0 & 6 & 7 \end{pmatrix}.$$

As all vertices in E_1 support cycles, all subsets of E_1^0 are saturated, and it is easy to identify the hereditary subsets of E_1^0. We notice that $C^*(E_1)$ is purely infinite and tight over the space $\mathcal{W}$ considered in Section 2.5. As the space $\mathcal{W}$ has the boundary decomposition property, it suffices to calculate $\mathrm{FK}_{\mathcal{B}}(C^*(E_1))$ by Theorem 3.3.4.

Recall that the invariant $\mathrm{FK}_{\mathcal{B}}$ consists of the groups and maps

$$\begin{tikzcd}
 & 3_1 \arrow[dr] & & 14_0 \\
1234_1 \arrow[ur] \arrow[r] \arrow[dr] & 2_1 \arrow[r] & 4_0 \arrow[ur] \arrow[r] \arrow[dr] & 24_0 \\
 & 1_1 \arrow[ur] & & 34_0
\end{tikzcd},$$

and $\mathrm{FK}_{\mathcal{B}}(C^*(E_1))$ is then

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dr, "\begin{pmatrix} 4 & 2 \end{pmatrix}"] & & \mathbb{Z}/2 \oplus \mathbb{Z}/48 \oplus \mathbb{Z} \\
\mathbb{Z}^3 \arrow[ur, "\begin{pmatrix} 0 \\ -12 \\ -1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"'] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 3 & 3 \end{pmatrix}"] & \mathbb{Z}/8 \oplus \mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 42 & 0 \\ 1 & 36 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 63 & 0 \\ 57 & 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 \end{pmatrix}"'] & \mathbb{Z}/72 \oplus \mathbb{Z} \\
 & \mathbb{Z} \arrow[ur, "\begin{pmatrix} 4 & 2 \end{pmatrix}"] & & \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}
\end{tikzcd}$$

by calculations similar to those in Example 3.3.3. Now consider the graph E_2 with adjacency matrix

$$A_{E_2} = \begin{pmatrix} 10 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 6 & 4 & 9 & 0 & 0 & 0 & 0 \\ 13 & 11 & 0 & 0 & 0 & 7 & 6 & 0 & 0 \\ 8 & 7 & 3 & 0 & 0 & 3 & 4 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 7 & 6 \\ 5 & 5 & 7 & 0 & 0 & 0 & 0 & 6 & 7 \end{pmatrix}.$$

Again, $C^*(E_2)$ is purely infinite and tight over $\mathcal{W}$. One can calculate that $\mathrm{FK}_{\mathcal{B}}(C^*(E_2))$ is

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dr, "\begin{pmatrix} 4 & 2 \end{pmatrix}"] & & \mathbb{Z}/2 \oplus \mathbb{Z}/48 \oplus \mathbb{Z} \\
\mathbb{Z}^3 \arrow[ur, "\begin{pmatrix} 0 \\ -12 \\ -1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"'] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 3 & 3 \end{pmatrix}"] & \mathbb{Z}/8 \oplus \mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 42 & 0 \\ 1 & 12 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 9 & 0 \\ 39 & 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}"'] & \mathbb{Z}/72 \oplus \mathbb{Z} \\
 & \mathbb{Z} \arrow[ur, "\begin{pmatrix} 4 & 2 \end{pmatrix}"] & & \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}
\end{tikzcd}$$

which is isomorphic to $\mathrm{FK}_{\mathcal{B}}(C^*(E_1))$. An isomorphism is given by the identity on the groups 1234_1, 3_1, 2_1, 1_1, and 4_0, together with the isomorphisms

$$\mathrm{FK}^0_{14}(C^*(E_1)) \xrightarrow{\begin{pmatrix} 1 & 24 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathrm{FK}^0_{14}(C^*(E_2))$$

$$\mathrm{FK}^0_{24}(C^*(E_1)) \xrightarrow{\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{FK}^0_{24}(C^*(E_2))$$

$$\mathrm{FK}^0_{34}(C^*(E_1)) \xrightarrow{\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \mathrm{FK}^0_{34}(C^*(E_2)).$$

By Theorem 3.3.4, Theorem 2.6.1, and Theorem 2.1.1, this isomorphism lifts to an isomorphism $C^*(E_1) \otimes \mathbb{K} \to C^*(E_2) \otimes \mathbb{K}$.

The graph E_2 was constructed from $\mathrm{FK}_{\mathcal{B}}(C^*(E_1))$ by first realising the K-theory of the simple subquotients 1, 2, 3, and 4 and then using the proof of Theorem 3.4.1 to construct 14, 24, and 34. Cf. the proof of Theorem 3.4.2 and the Theorems 3.3.4 and 3.3.5.

EXAMPLE 3.3.7. Consider the graph E_1 with adjacency matrix

$$A_{E_1} = \begin{pmatrix} 5 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 7 & 6 & 0 & 0 \\ 1 & 1 & 0 & 0 & 6 & 7 & 0 & 0 \\ 0 & 0 & 6 & 4 & 6 & 3 & 9 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 & 9 \end{pmatrix}.$$

As all vertices in E_1 support cycles, all subsets of E_1^0 are saturated, and it is easy to identify the hereditary subsets of E_1^0. We notice that $C^*(E_1)$ is purely infinite and tight over the space $\mathcal{D}$ considered in Section 2.5. By Theorem 3.2.4, it suffices to calculate $\mathrm{FK}_{\mathcal{R}}(C^*(E_1))$. Recall that $\mathrm{FK}_{\mathcal{R}}$ for $\mathcal{D}$ consists of the groups and maps

$$\begin{tikzcd}
 & 3_1 \arrow[dr] & & 24_0 \arrow[dr] & & \\
4_1 & & 4_0 \arrow[ur] \arrow[dr] & 1_1 \arrow[r] & 234_0 \arrow[r] & 1234_0 \\
 & 2_1 \arrow[ur] & & 34_0 \arrow[ur] & &
\end{tikzcd}.$$

Then the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}(C^*(E_1))$ of $C^*(E_1)$ is

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, "\begin{pmatrix} 3 & 3 \end{pmatrix}"'] & & \mathbb{Z}/2 \oplus \mathbb{Z}/24 \oplus \mathbb{Z} \arrow[drr, "\begin{pmatrix} 0 & 6 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}"] & & & \\
 & \mathbb{Z}/4 \oplus \mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 18 & 0 \\ 1 & 12 & 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 9 & 0 \\ 3 & 0 \end{pmatrix}"'] & & \mathbb{Z} \arrow[r, "\begin{pmatrix} 1 & 9 & 2 & 3 \end{pmatrix}"] & \mathbb{Z}/3 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 & 64 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 76 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}"] & \mathbb{Z}/3 \oplus \mathbb{Z}/96 \oplus \mathbb{Z}^2 \\
\mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 2 \end{pmatrix}"] & & \mathbb{Z}/36 \oplus \mathbb{Z} \arrow[urr, "\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}"'] & & &
\end{tikzcd}$$

together with $\mathrm{FK}_4^1(C^*(E)) = \mathbb{Z}$. Now consider the graph E_2 with adjacency matrix

$$A_{E_2} = \begin{pmatrix} 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 7 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 7 & 9 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 13 & 6 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 12 & 7 & 0 & 0 & 0 \\ 26 & 27 & 6 & 6 & 12 & 6 & 10 & 8 & 8 \\ 22 & 25 & 6 & 16 & 11 & 6 & 8 & 9 & 8 \\ 15 & 16 & 6 & 18 & 12 & 5 & 8 & 8 & 9 \end{pmatrix}.$$

Again, $C^*(E_2)$ is purely infinite and tight over $\mathcal{D}$. One can calculate that $\mathrm{FK}_{\mathcal{R}}(C^*(E_2))$ is

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, "\begin{pmatrix} 3 & 3 \end{pmatrix}"'] & & \mathbb{Z}/2 \oplus \mathbb{Z}/24 \oplus \mathbb{Z} \arrow[dr, "\begin{pmatrix} 0 & 6 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}"] & & \\
& \mathbb{Z}/4 \oplus \mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 18 & 0 \\ 1 & 0 & 0 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 27 & 0 \\ 21 & 0 \end{pmatrix}"'] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 0 & 5 & 2 & 3 \end{pmatrix}"] & \mathbb{Z}/3 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 2 & 32 & 0 & 0 \\ 1 & 56 & 0 & 0 \\ 2 & 76 & -3 & 0 \\ 2 & 80 & 2 & 0 \end{pmatrix}"] & \mathbb{Z}/3 \oplus \mathbb{Z}/96 \oplus \mathbb{Z}^2 \\
\mathbb{Z} \arrow[ur, "\begin{pmatrix} 0 & 2 \end{pmatrix}"] & & \mathbb{Z}/36 \oplus \mathbb{Z} \arrow[ur, "\begin{pmatrix} 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}"'] & &
\end{tikzcd}$$

together with $\mathrm{FK}_4^1(C^*(E_2)) = \mathbb{Z}$. An isomorphism $\mathrm{FK}_{\mathcal{R}}(C^*(E_1)) \to \mathrm{FK}_{\mathcal{R}}(C^*(E_2))$ is given by the identity on the groups 4_1, 3_1, 2_1, and 4_0, together with the isomorphisms

$$\mathrm{FK}_{24}^0(C^*(E_1)) \xrightarrow{\begin{pmatrix} 1 & 12 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathrm{FK}_{24}^0(C^*(E_2))$$

$$\mathrm{FK}_{34}^0(C^*(E_1)) \xrightarrow{\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{FK}_{34}^0(C^*(E_2))$$

$$\mathrm{FK}_{234}^0(C^*(E_1)) \xrightarrow{\begin{pmatrix} 0 & 8 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \mathrm{FK}_{234}^0(C^*(E_2))$$

$$\mathrm{FK}_{1234}^0(C^*(E_1)) \xrightarrow{\begin{pmatrix} 1 & 64 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \mathrm{FK}_{1234}^0(C^*(E_2)).$$

By Theorem 3.2.4, Theorem 2.5.4, and Theorem 2.1.1, this isomorphism lifts to an isomorphism $C^*(E_1) \otimes \mathbb{K} \to C^*(E_2) \otimes \mathbb{K}$.

The graph E_2 was constructed from $\mathrm{FK}_{\mathcal{R}}(C^*(E_1))$ by first realising the K-theory of the simple subquotients 1, 2, 3, and 4, and then using the proof of Theorem 3.4.1 to construct first 24 and 34, and then 1234, as in the proof of Theorem 3.4.2.

3.4. Range of filtered K-theory for graph algebras

For a real rank zero graph algebra A, its filtered K-theory $\mathrm{FK}(A)$ has the properties that $K_1(A(Y))$ is free for all $Y \in \mathbb{LC}(X)$, as $A(Y)$ is a graph algebra, and that the map $K_0(A(Y \setminus U)) \to K_1(A(U))$ vanishes for all $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$, as A has real rank zero.

One could then ask if any exact $\mathcal{NT}$-module satisfying these two conditions is the filtered K-theory of some real rank zero graph algebra.

In [**ABK**], the range of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ is determined for graph algebras, so it can be concluded by Theorem 3.4.2 combined with Theorem 3.3.5 and the proof of Theorem 3.2.4, that for accordion spaces and for the spaces $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{D}$, and $\mathcal{S}$, the answer to the above question is yes. The construction uses a result of S. Eilers, T. Katsura, M. Tomforde, and J. West, dealing with extensions.

THEOREM 3.4.1 ([**EKTW**, 4.3, 4.7]). *Let $\mathcal{E}$*

$$\begin{tikzcd} G_1 \arrow[r, "\varepsilon"] & G_2 \arrow[r, "\gamma"] & G_3 \arrow[d, "0"] \\ F_3 \arrow[u, "\delta"] & F_2 \arrow[l, "\gamma'"] & F_1 \arrow[l, "\varepsilon'"] \end{tikzcd}$$

be an exact sequence of abelian groups with F_1, F_2, F_3 free. Suppose that there exists row-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{1, 2, \dots, \infty\}$ with isomorphisms

$$\alpha_1\colon \operatorname{coker} A \to G_1, \quad \beta_1\colon \ker A \to F_1,$$

$$\alpha_3\colon \operatorname{coker} B \to G_3, \quad \beta_3\colon \ker B \to F_3.$$

Then there exists a row-finite matrix $Y \in M_{n_3,n_1'}(\mathbb{Z})$ and isomorphisms

$$\alpha_2\colon \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to G_2, \quad \beta_2\colon \ker\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to F_2$$

such that $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ gives an isomorphism of complexes from the exact sequence

$$\begin{tikzcd} \operatorname{coker} A \arrow[r, "I"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[r, "P"] & \operatorname{coker} B \arrow[d, "0"] \\ \operatorname{coker} B \arrow[u, "{[Y]}"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[l, "P'"] & \operatorname{coker} A \arrow[l, "I'"] \end{tikzcd}$$

where the maps I, I' and P, P' are induced by the obvious inclusions or projections, to the exact sequence $\mathcal{E}$.

If there exist an $A' \in M_{n_1',n_1}$ *such that* $A'A - 1 \in M_{n_1',n_1'}(\mathbb{Z}_+)$*, then* Y *can be chosen such that* $Y \in M_{n_3,n_1'}(\mathbb{Z}_+)$*. If furthermore a row-finite matrix* $Z \in M_{n_3,n_1'}(\mathbb{Z})$ *is given, then* Y *can be chosen such that* $Y - Z \in M_{n_3,n_1'}(\mathbb{Z}_+)$*.*

For a graph E_1 with adjacency matrix $A+1$, and a graph E_3 with adjacency matrix $B + 1$, the matrix Y describes how edges should be added from vertices in E_1 to vertices in E_3 to form a graph E_2 with adjacency matrix $\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} + 1$, such that $C^*(E_1)$ is stably isomorphic to an ideal in $C^*(E_2)$ with quotient $C^*(E_3)$, and such that the desired six-term sequence is induced by the extension.

Theorem 3.4.2 ([**ABK**, 9.2]). *Let* A *be a* C^**-algebra over* X *with* $K_1(A(x))$ *free for all* $x \in X$*. Then there exists a countable, directed graph* E *with the property that all vertices in* E *are regular and support at least two cycles, and that* $C^*(E)$ *is tight over* X *and has* $\mathrm{FK}_{\mathcal{R}}(C^*(E))$ *isomorphic to* $\mathrm{FK}_{\mathcal{R}}(A)$*.*

The graph E *can be chosen to be finite if* $K_1(A(x))$ *and* $K_0(A(\widetilde{\{x\}}))$ *are finitely generated, and the rank of* $K_1(A(x))$ *coincides with the rank of the cokernel of* $i\colon K_0(A(\widetilde{\partial}(x))) \to K_0(A(\widetilde{\{x\}}))$*, for all* $x \in X$*.*

Notice that the constructed graph algebra $C^*(E)$ is purely infinite, and that it is a Cuntz-Krieger algebra if E is finite.

Sketch of proof. The idea is to realize the simple subquotients as simple, purely infinite graph algebras E_x and then apply Theorem 3.4.1 recursively.

Let $x \in X$ and assume that for all $y, z \in \widetilde{\partial}(x)$, the vertices in E_y and E_z have already been connected with edges if needed. By exactness, the resulting graph $E_{\widetilde{\partial}(x)}$ has $K_0(C^*(E_{\widetilde{\partial}(x)})) \cong \mathrm{FK}^0_{\widetilde{\partial}(x)}(A)$, so by applying Theorem 3.4.1 on

$$K_1(C^*(E_x)) \cong \mathrm{FK}^1_{\{x\}}(A) \to \mathrm{FK}^0_{\widetilde{\partial}(x)}(A) \to \mathrm{FK}_{\widetilde{\{x\}}}(A) \to K_0(C^*(E_x)),$$

edges are added for all $y \in \widetilde{\partial}(x)$ from E_y to E_x such that $\mathrm{FK}^0_{\widetilde{\{x\}}}(A)$ is realized.

To assure that $C^*(E)$ is tight over X, the graphs E_x are chosen such that all vertices are regular and support at least two cycles, hence ideals in $C^*(E)$ correspond to hereditary subsets in E^0, and by Theorem 3.4.1 we can make the construction such that there are edges from vertices in E_y to vertices in E_x exactly when $y > x$. ☐

3.4.1. Extensions of Cuntz-Krieger algebras.

For AF algebras, an extension of AF algebras is always an AF algebra. For an extension $I \hookrightarrow A \twoheadrightarrow A/I$ of real rank zero C^*-algebras I and A/I, the C^*-algebra A has real rank zero if and only if the induced map on K-theory $K_0(A/I) \to K_1(I)$ vanishes. And for an extension $I \hookrightarrow A \twoheadrightarrow A/I$ of stable rank one C^*-algebras I and A/I, the C^*-algebra A has stable rank one if and only if the induced map on K-theory $K_1(A/I) \to K_0(I)$ vanishes.

It is desirable to establish a similar result for real rank zero Cuntz-Krieger algebras, i.e., that given an extension $I \hookrightarrow A \twoheadrightarrow A/I$ of stabilized real rank zero Cuntz-Krieger algebras I and A/I, the C^*-algebra A is a stabilized real rank zero Cuntz-Krieger algebra if and only if some condition on the level of K-theory is satisfied.

Knowing the range of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ (cf. Theorem 3.4.2) and that it is a complete invariant, we can establish such a result. As we restrict to Cuntz-Krieger algebras of real rank zero, the condition that $K_0(A/I) \to K_1(I)$ vanishes is necessary. It turns out to be sufficient, provided the primitive ideal space is an accordion space or homeomorphic to one of the spaces $\mathcal{W}$, $\mathcal{W}^{\mathrm{op}}$, $\mathcal{Y}$, $\mathcal{Y}^{\mathrm{op}}$, and $\mathcal{D}$, since such an extension would otherwise be a phantom Cuntz-Krieger algebra.

COROLLARY 3.4.3 ([**ABK**, 9.5]). *Let X be a finite T_0-space and assume that $\mathrm{FK}_{\mathcal{R}}$ is a complete invariant for real rank zero, purely infinite, nuclear, separable C^*-algebras that are tight over X and satisfy the property that for all $x \in X$, $A(x)$ is in the bootstrap class and $K_1(A(x))$ is free.*

Let $I \hookrightarrow A \twoheadrightarrow B$ be an extension of C^*-algebras where A has primitive ideal space X. Then A is stably isomorphic to a real rank zero Cuntz-Krieger algebra if and only if I and A/I are stably isomorphic to real rank zero Cuntz-Krieger algebras, and the induced map $K_0(A/I) \to K_1(I)$ vanishes.

Articles

Here follows first the article *Filtrated K-theory of real rank zero C^*-algebras*, [**ARR**], which is written with G. Restorff and E. Ruiz and is to appear in *International Journal of Mathematics*, and then the article *Reduction of filtered K-theory and a characterization of Cuntz-Krieger algebras*, [**ABK**], which is written with R. Bentmann and T. Katsura.

Please note that in the article *Filtrated K-theory of real rank zero C^*-algebras*, the term "filtrated K-theory" is used instead of the term "filtered K-theory". The authors made this choice since the subject of the article is a question that was raised in response to the article *C^*-algebras over topological spaces: Filtrated K-theory*, [**MN**], by R. Meyer and R. Nest.

FILTRATED K-THEORY FOR REAL RANK ZERO C^*-ALGEBRAS

SARA ARKLINT, GUNNAR RESTORFF, AND EFREN RUIZ

Abstract. Using Kirchberg KK_X-classification of stable, purely infinite, nuclear, separable C^*-algebras with finite primitive ideal space, Bentmann showed that filtrated K-theory classifies stable, purely infinite, nuclear, separable C^*-algebras that satisfy that all simple subquotients are in the bootstrap class and that the primitive ideal space is finite and of a certain type, referred to as accordion spaces. This result generalizes the results of Meyer-Nest involving finite linearly ordered spaces. Examples have been provided, for any finite non-accordion space, that isomorphic filtrated K-theory does not imply KK_X-equivalence for this class of C^*-algebras. As a consequence, for any non-accordion space, filtrated K-theory is not a complete invariant for stable, purely infinite, nuclear, separable C^*-algebras that satisfy that all simple subquotients are in the bootstrap class.

In this paper, we investigate the case for real rank zero C^*-algebras and four-point primitive ideal spaces, as this is the smallest size of non-accordion spaces. Up to homeomorphism, there are ten different connected T_0-spaces with exactly four points. We show that filtrated K-theory classifies real rank zero, stable, purely infinite, nuclear, separable C^*-algebras that satisfy that all simple subquotients are in the bootstrap class for eight out of ten of these spaces.

1. Introduction

The C^*-algebra classification programme initiated by G. A. Elliott in the early seventies has seen a rapid development during the past 20 years. The notion of real rank zero C^*-algebras introduced by G. K. Pedersen and L. G. Brown in the late eighties has turned out to be of particular interest in connection with classification of C^*-algebras. Until the mid-nineties most results were concerned with the stably finite algebras, when people such as M. Rørdam, N. C. Phillips, E. Kirchberg and D. Huang classified some purely infinite, nuclear, separable C^*-algebras in the bootstrap class. All these had finitely many ideals — in fact, almost all cases were either the simple case or the one non-trivial ideal case. D. Huang was also able to classify purely infinite Cuntz-Krieger algebras with finite K-theory (implying that all the K_1-groups are zero). In contrast to the stably finite case, the positive cone of purely infinite C^*-algebras carries no extra information, so it was clear from the beginning, that to classify non-simple purely infinite C^*-algebras one needs to come up with a new invariant, which also encodes the ideal structure and the K-theory of all ideals and quotients.

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The main ingredients of the proof of N. C. Phillips and E. Kirchberg were the UCT of J. Rosenberg and C. Schochet and a result saying that every KK-equivalence between (simple, purely infinite, stable, nuclear, separable) C^*-algebras can be lifted to a $*$-isomorphism between the algebras. Shortly after, E. Kirchberg generalized this result to X-equivariant KK-theory, where X is (homeomorphic to) the primitive ideal space of the C^*-algebra. The only ingredient thus missing to classify purely infinite, nuclear, separable, stable C^*-algebras seemed to be to find the right invariant and prove a UCT for X-equivariant KK-theory with this new invariant. For the case with one non-trivial ideal, A. Bonkat reproved Rørdams result by providing a UCT for this class using the cyclic six-term exact sequence in K-theory. The second named author generalized this to two non-trivial ideals by including four cyclic six-term exact sequences. R. Meyer and R. Nest, and R. Bentmann recently proved that the obvious guess of an invariant gives a UCT for certain ideal lattices — the so-called accordion spaces (including, e.g., all C^*-algebras with exactly three primitive ideals). In turn they also provide a series of counter-examples, where we do not have a UCT. They actually find examples of stable, purely infinite, nuclear, separable C^*-algebras in the bootstrap class with finitely many ideals having isomorphic invariants without being isomorphic. This result seems to be in sharp contrast to the stable classification result for all purely infinite Cuntz-Krieger algebras with finitely many ideals obtained by the second named author by use of methods from shift spaces.

We find it very likely that the reason that Cuntz-Krieger algebras are classifiable, is the restrictive nature of their K-theory. In this paper we examine what happens to real rank zero algebras in the cases where the primitive ideal space has exactly four points. Moreover, we assume that the space is connected (since otherwise the algebras are direct sums of algebras with fewer than four primitive ideals). Also, all the basic counterexamples of R. Meyer, R. Nest, and R. Bentmann are formulated for algebras with four primitive ideals. Up to homeomorphism, there are ten different connected T_0-spaces with exactly four points. These are

$$\begin{aligned}
\mathcal{O}(X_1) &= \{\emptyset, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_2) &= \{\emptyset, \{4\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_3) &= \{\emptyset, \{4\}, \{3,4\}, \{2,4\}, \{2,3,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_4) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_5) &= \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_6) &= \{\emptyset, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_7) &= \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_8) &= \{\emptyset, \{1\}, \{4\}, \{1,2\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_9) &= \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\},\\
\mathcal{O}(X_{10}) &= \{\emptyset, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}.
\end{aligned}$$

R. Meyer and R. Nest, and R. Bentmann have proved that the spaces X_7, X_8, X_9 and X_{10} have a UCT, and thus we can classify stable, purely infinite, nuclear, separable C^*-algebras in the bootstrap class with these spaces as primitive ideal spaces. Moreover they have provided counter-examples for classification for all the spaces $X_1, X_2, X_3, X_4, X_5, X_6$. In this paper we prove the following

Theorem 1.1. *Let A and B be purely infinite, nuclear, separable C^*-algebras of real rank zero in the bootstrap class of R. Meyer and R. Nest (cf.* [MN09, 4.11]*). Assume that the primitive ideal space of A and B both are homeomorphic to X_i for an $i = 1, 2, 4, 5, 7, 8, 9, 10$.*

1. *If A and B are stable, then every isomorphism from* $\mathrm{FK}(A)$ *to* $\mathrm{FK}(B)$ *can be lifted to a $*$-isomorphism from A to B.*
2. *If A and B are unital, then every isomorphism from* $\mathrm{FK}(A)$ *to* $\mathrm{FK}(B)$ *that preserves the unit can be lifted to a $*$-isomorphism from A to B.*

Theorem 1.2. *There exist stable, purely infinite, nuclear, separable C^*-algebras of real rank zero in the bootstrap class of R. Meyer and R. Nest (cf.* [MN09, 4.11]*) with the primitive ideal space homeomorphic to X_3, which have isomorphic filtrated* K*-theory without being isomorphic.*

where FK denotes the functor filtrated K-theory which will be defined shortly.

For the case where the primitive ideal space is isomorphic to X_6 there are still no counterexamples for the real rank zero case — however our methods do not apply as there is no known finite refinement of FK which gives a UCT.

In general the unital part of Theorem 1.1 follows from the stable part by using results from [RR07]. For X_7, Theorem 1.1 is proved by R. Meyer and R. Nest in [MN, 4.14], for X_8, X_9 and X_{10}, it is proved by R. Bentmann in [Ben10, 5.4.2]. In Section 2 of this paper we set up notation and prove some preliminary results used later in this paper. In Sections 3 and 4 Theorem 1.1 is proved for X_1, X_2, X_4 and X_5 (cf. Corollaries 3.9 and 4.6 and Remarks 3.10 and 4.7). The proofs rely on the result [Kir00, 4.3] of E. Kirchberg that $\mathrm{KK}(X)$-equivalences lift to X-equivariant isomorphisms for stable, purely infinite, nuclear, separable C^*-algebras with primitive ideal space homeomorphic to a finite T_0-space X. Theorem 1.2 is proved in Section 5.

2. Preliminaries and notation

In this section, we briefly discuss C^*-algebras over a topological space X and the invariant introduced by R. Meyer and R. Nest in [MN] called filtrated K-theory. We refer the reader to [MN] for details.

We would like to note that there are other invariants in the literature which are closely related to filtrated K-theory. Examples are filtered K-theory and ideal related K-theory. It has been proved by R. Meyer and R. Nest in [MN] and R. Bentmann in [Ben10, 5.4.2] that for the spaces X_i that these invariants are naturally isomorphic to filtrated K-theory. It is not known if these invariants are naturally isomorphic for all finite topological spaces.

2.1. **C^*-algebras over a topological space X.** A C^*-algebra over a topological space X is a pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \mathrm{Prim}(A) \to X$ where $\mathrm{Prim}(A)$ denotes the primitive ideal space of A. Assume from now on that X is a finite topological space satisfying the T_0 separation axiom, i.e., such that $\overline{\{x\}} \neq \overline{\{y\}}$ for all $x, y \in X$ with $x \neq y$. Let $\mathcal{O}(X)$ denote the open subsets of X, and let $\mathbb{I}(A)$ denote the lattice of (two-sided, closed) ideals of A. A C^*-algebra over X can then equivalently be defined as a pair (A, ψ) consisting of a C^*-algebra A and a map $\psi\colon \mathcal{O}(X) \to \mathbb{I}(A)$ that preserves infima and suprema. We then write $A(U)$ for $\psi(U)$.

The locally closed subsets of X are denoted by $\mathbb{LC}(X) = \{U \backslash V \mid V, U \in \mathcal{O}(X), V \subseteq U\}$, and the connected, non-empty, locally closed subsets of X are denoted by $\mathbb{LC}(X)^*$. For $Y \in \mathbb{LC}(X)$ we define $A(Y) = A(U)/A(V)$ when $Y = U \backslash V$ for some $V, U \in \mathcal{O}(X)$ satisfying $V \subseteq U$. Up to canonical isomorphism, $A(Y)$ does not depend on the choice of U and V.

For C^*-algebras A and B over X, we say that a $*$-homomorphism $\varphi\colon A \to B$ is X-equivariant if $\varphi(A(U)) \subseteq B(U)$ holds for all $U \in \mathcal{O}(X)$. An extension $A \hookrightarrow B \twoheadrightarrow C$ is called X-equivariant if it induces an extension $A(U) \hookrightarrow B(U) \twoheadrightarrow C(U)$ for all $U \in \mathcal{O}(X)$.

E. Kirchberg has constructed X-equivariant KK-theory $\mathrm{KK}_*(X; -, -)$, also called ideal related KK-theory and here referred to as $\mathrm{KK}(X)$-theory. We denote by $\mathfrak{KK}(X)$ the category of separable C^*-algebras over X with $\mathrm{KK}_0(X)$-classes as morphism groups. In [MN09, 3.11], R. Meyer and R. Nest show that the category $\mathfrak{KK}(X)$ equipped with the suspension automorphism S and mapping cone triangles as distinguished triangles is triangulated; so mapping cones of X-equivariant $*$-homomorphisms give exact triangles, and so do extensions over X that split by an X-equivariant completely positive contraction.

2.2. **Filtrated K-theory FK and the UCT.** One defines for each $Y \in \mathcal{O}(X)$ the functor FK_Y by $\mathrm{FK}_Y(A) = \mathrm{K}_*(A(Y))$. We write $\mathrm{FK}_Y^i(A)$ for $\mathrm{K}_i(A(Y))$. In [MN] R. Meyer and R. Nest construct commutative C^*-algebras R_Y over X such that $\mathrm{KK}_*(X; R_Y, -)$ and FK_Y are equivalent functors.

By the Yoneda Lemma, cf. [ML98, 3.2], the set $\mathcal{NT}(Y, Z)$ of natural transformations from the functor FK_Y to the functor FK_Z is then given by $\mathrm{KK}_*(X; R_Z, R_Y)$. Given $\alpha \in \mathrm{KK}_*(X; R_Z, R_Y)$ we denote by $\bar{\alpha}$ the corresponding element in $\mathcal{NT}(Y, Z)$ given by $\alpha \boxtimes -$ where $- \boxtimes -$ denotes the X-equivariant Kasparov product. Given $f \in \mathcal{NT}(Y, Z)$, we let $\hat{f}$ denote the corresponding element in $\mathrm{KK}_*(X; R_Z, R_Y)$.

The functor FK is then defined as the family of functors $(\mathrm{FK}_Y)_{Y \in \mathbb{LC}(X)^*}$ together with the sets $\mathcal{NT}(Y, Z)$ of natural transformations. The target category of FK is the category of modules over the ring $\mathcal{NT} = \bigoplus_{Y, Z \in \mathbb{LC}(X)^*} \mathcal{NT}(Y, Z)$. A homomorphism $\mathrm{FK}(A) \to \mathrm{FK}(B)$ is then a family of homomorphisms (φ_Y) that respects the natural transformations. Kasparov multiplication induces a map $\mathrm{KK}_*(X; A, B) \to \mathrm{Hom}(\mathrm{FK}(A), \mathrm{FK}(B))$, and for $A = R_Y$ this map is an isomorphism.

In [MN] R. Meyer and R. Nest establish a UCT for $\mathrm{KK}(X)$-theory, i.e., they establish exactness of

$$\mathrm{Ext}^1_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \hookrightarrow \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$$

for A and B separable C^*-algebras over X with A belonging to the bootstrap class $\mathcal{B}(X)$ defined by R. Meyer and R. Nest, cf. [MN09, 4.11], and with $\mathrm{FK}(A)$ having projective dimension at most 1 as a module over $\mathcal{NT}$. By construction, $\mathrm{FK}(R_Y)$ has projective dimension 0 for all $Y \in \mathbb{LC}(X)^*$. By [MN09, 4.13], a nuclear C^*-algebra over X belongs to $\mathcal{B}(X)$ if and only if its simple subquotients belong to the bootstrap class of J. Rosenberg and C. Schochet.

2.3. **Construction of R_Y.** The C^*-algebras R_Y are constructed as follows. Define a partial order on X by $x \leq y$ when $x \in \overline{\{y\}}$. The order complex $\mathrm{Ch}(X)$ is the geometric realisation of the simplicial set whose nondegenerate n-simplices $[x_0, \dots, x_n]$ are strict chains $x_0 < \cdots < x_n$. Maps $m, M\colon \mathrm{Ch}(X) \to X$ are then defined by the inner of a simplex $[x_0, \ldots, x_n]$ being sent to x_0 by m and to x_n by M. The C^*-algebras R_Y over X are then defined by $R_Y(Z) = C_0(m^{-1}(Y) \cap M^{-1}(Z))$ for all $Y, Z \in \mathbb{LC}(X)$.

For $Y \in \mathbb{LC}(X)$ and $U \in \mathcal{O}(Y)$, we then get X-equivariant extensions $R_{Y\setminus U} \hookrightarrow R_Y \twoheadrightarrow R_U$. The natural transformation given by $R_{Y\setminus U} \hookrightarrow R_Y$ is denoted $r_Y^{Y\setminus U}$ and called a restriction map, the natural transformation given by $R_Y \twoheadrightarrow R_U$ is denoted by i_U^Y and called an extension map, and the natural transformation given by $R_{Y\setminus U} \hookrightarrow R_Y \twoheadrightarrow R_U$ is denoted by $\delta_{Y\setminus U}^U$ and called a boundary map. For a C^*-algebra A over X, these natural transformations are the ones appearing in the six-term exact sequence induced by the extension $A(U) \hookrightarrow A(Y) \twoheadrightarrow A(Y\setminus U)$. It is unknown whether there exists finite T_0-spaces X over which the ring $\mathcal{NT}$ is not generated by transformations of this form, but for the spaces $X_1, X_2, \ldots, X_{10}$ considered in this paper, this is not the case.

3. The counterexample of Meyer and Nest

We now restrict to the space $X_1 = \{1, 2, 3, 4\}$ with $\mathcal{O}(X_1) = \{\emptyset\} \cup \{U \subseteq X_1 \mid 4 \in U\}$. We abbreviate, e.g., $\{1, 2, 3\}$ to 123. A C^*-algebra A over X_1 is then an extension of the form $A(4) \hookrightarrow A \twoheadrightarrow A(1) \oplus A(2) \oplus A(3)$. The ordering on X induced by its topology is then defined by $i \leq 4$ for all $i \in X_1$, its Hasse diagram (or, more correctly, the Hasse diagram of the inverse order relation) is

$$\begin{tikzcd}
1 & 2 & 3 \\
& 4 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd},$$

and $\mathbb{LC}(X_1)^* = \{4, 14, 24, 34, 124, 134, 234, 1234, 1, 2, 3\}$. In [MN] it is shown that the ring $\mathcal{NT} = \bigoplus_{Y,Z \in \mathbb{LC}(X_1)^*} \mathcal{NT}(Y, Z)$ is generated by natural transformations i, r and δ that are induced by six-term exact sequences, and the indecomposable transformations are of infinite order and fit into the following diagram

$$\begin{tikzcd}
& 14 \arrow[r, "i"] \arrow[dr, "i"] & 124 \arrow[dr, "i"] & & 1 \arrow[dr, "\delta", "\circ" marking] & \\
4 \arrow[ur, "i"] \arrow[r, "i"] \arrow[dr, "i"] & 24 \arrow[ur, "i"] \arrow[dr, "i"] & 134 \arrow[r, "i"] & 1234 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "r"] & 2 \arrow[r, "\delta", "\circ" marking] & 4 \\
& 34 \arrow[ur, "i"] \arrow[r, "i"] & 234 \arrow[ur, "i"] & & 3 \arrow[ur, "\delta", "\circ" marking] &
\end{tikzcd}$$

where the six squares commute and the sum of the three transformations from 1234 to 4 vanishes.

3.1. **The refined invariant.** In [MN], R. Meyer and R. Nest refine the invariant FK to an invariant FK′. They prove a UCT for this refined invariant, so for A and B in the bootstrap class $\mathcal{B}(X_1)$ one can lift isomorphisms between $\mathrm{FK}'(A)$ and $\mathrm{FK}'(B)$ to $\mathrm{KK}(X_1)$-equivalences, and by combining this with the classification result [Kir00, 4.3] of E. Kirchberg conclude that it strongly classifies the stable, purely infinite, separable, nuclear C^*-algebras A that are tight over X_1 and whose simple subquotients $A(4)$, $A(1)$, $A(2)$ and $A(3)$ lie in the bootstrap class, see [MN, 5.14, 5.15].

In [RR07], the second and third author showed how one can strongly classify a class of unital properly infinite C^*-algebras given that the this class are strongly classified up to stable isomorphism. Since $\mathrm{FK}'(\cdot)$ strongly classifies the class of stable, purely infinite, separable, nuclear C^*-algebras A that are tight over X_1, by Theorem 2.1 of [RR07], $\mathrm{FK}'(\cdot)$ together with class of the unit strongly classifies the class of unital, purely infinite, separable, nuclear C^*-algebras A that are tight over X_1.

The invariant is defined by constructing a C^*-algebra R_{12344} over X_1 and adding $\mathrm{KK}_*(X_1; R_{12344}, -)$ to the family of functors. The indecomposable transformations in the larger ring $\mathcal{NT}' = \bigoplus_{Y,Z \in \mathbb{LC}(X_1)^* \cup \{12344\}} \mathcal{NT}(Y,Z)$ fit into the following diagram:

$$\begin{tikzcd}
 & 14 \arrow[dr] & & 124 \arrow[dr, "i"] & & 1 \arrow[dr, "\delta", "\circ" marking] & \\
4 \arrow[ur, "i"] \arrow[r, "i"] \arrow[dr, "i"'] & 24 \arrow[r] & 12344 \arrow[ur] \arrow[r] \arrow[dr] & 134 \arrow[r, "i"] & 1234 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "r"'] & 2 \arrow[r, "\delta", "\circ" marking] & 4 \\
 & 34 \arrow[ur] & & 234 \arrow[ur, "i"] & & 3 \arrow[ur, "\delta", "\circ" marking] &
\end{tikzcd} \tag{3.1}$$

The C^*-algebra R_{12344} is the mapping cone of a generator of the cyclic free group $\mathcal{NT}(234, 14)$ and its filtrated K-theory is

$$\begin{tikzcd}
 & 0 \arrow[r, "i"] \arrow[dr, "i"'] & \mathbb{Z} \arrow[dr, "i"] & & \mathbb{Z} \arrow[dr, "\delta", "\circ" marking] & \\
\mathbb{Z}[1] \arrow[ur, "i"] \arrow[r, "i"] \arrow[dr, "i"'] & 0 \arrow[ur, "i"] \arrow[dr, "i"'] & \mathbb{Z} \arrow[r, "i"] & \mathbb{Z}^2 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "r"'] & \mathbb{Z} \arrow[r, "\delta", "\circ" marking] & \mathbb{Z}[1] \\
 & 0 \arrow[r, "i"] \arrow[ur, "i"] & \mathbb{Z} \arrow[ur, "i"] & & \mathbb{Z} \arrow[ur, "\delta", "\circ" marking] &
\end{tikzcd} \tag{3.2}$$

where the three maps i_{ij4}^{1234} are given by the three coordinate embeddings $\mathbb{Z} \to \mathbb{Z}^3/(1,1,1)$, the three maps r_{1234}^k are given by the three projections $\mathbb{Z}^3/(1,1,1) \to \mathbb{Z}^2/(1,1)$ onto coordinate hyperplanes, and the three maps δ_k^4 are the identity.

Lemma 3.1. *Assume that* $\mathrm{FK}_Y(A)$ *and* $\mathrm{FK}_Y(R_{12344})$ *are isomorphic for all* $Y \in \mathbb{LC}(X_1)^*$ *and that* $i_{124}^{1234} \oplus i_{134}^{1234} \colon \mathrm{FK}_{124}(A) \oplus \mathrm{FK}_{134}(A) \to \mathrm{FK}_{1234}(A)$ *is an isomorphism. Then* $\mathrm{FK}(A)$ *and* $\mathrm{FK}(R_{12344})$ *are isomorphic as* $\mathcal{NT}$*-modules and* A *and* R_{12344} *are* $\mathrm{KK}(X_1)$*-equivalent.*

Proof. Define for each $Y \in \mathbb{LC}(X)^*$ an $\mathcal{NT}$-module P_Y as $P_Y(Z) = \mathcal{NT}(Y,Z)$. Then P_Y is freely generated by $\mathrm{id}_Y \in P_Y(Y)$ as an $\mathcal{NT}$-module. Define $j \colon P_{1234} \to P_{124} \oplus P_{134} \oplus P_{234}$ by $f \mapsto f i_{124}^{1234} + f i_{134}^{1234} + f i_{234}^{1234}$. Then $\mathrm{FK}(R_{12344})$ is isomorphic to $\operatorname{coker} j$ as $\mathcal{NT}$-modules, cf. [MN, Section 5], with $\operatorname{im} j$ generated by $i_{124}^{1234} + i_{134}^{1234} + i_{234}^{1234}$.

Hence an $\mathcal{NT}$-morphism $\mathrm{FK}(R_{12344}) \to \mathrm{FK}(A)$ can be defined by choosing elements $g_Y \in \mathrm{FK}_Y(A)$, $Y \in \{124, 134, 234\}$, satisfying $i_{124}^{1234}(g_{124}) + i_{134}^{1234}(g_{134}) + i_{234}^{1234}(g_{234}) = 0$, and defining the map by $\mathrm{id}_Y \mapsto g_Y$ for $Y \in \{124, 134, 234\}$ and expanding by $\mathcal{NT}$-linearity.

If g_Y generates $\mathrm{FK}_Y(A)$ for all $Y \in \{124, 134, 234\}$, then the morphism will be an isomorphism: it is automatically bijective $\mathrm{FK}_Z(R_{12344}) \to \mathrm{FK}_Z(A)$ for $Z \in \{124, 134, 234\}$, by the assumptions in the lemma it is therefore surjective and hence bijective for $Z = 1234$, and by exactness it then follows that it is bijective for $Z \in \{1, 2, 3\}$ whereby bijectivity for $Z = 4$ also follows, cf. the Diagram (3.2).

Let g_Y be a generator of $\mathrm{FK}_Y(A)$ for $Y \in \{124, 134, 234\}$. Since $i_{124}^{1234} \oplus i_{134}^{1234}$ is an isomorphism, $\mathrm{FK}_{1234}(A)$ is spanned by $i_{124}^{1234}(g_{124})$ and $i_{134}^{1234}(g_{134})$ so we may write $i_{234}^{1234}(g_{234}) = mi_{124}^{1234}(g_{124}) + ni_{134}^{1234}(g_{134})$ for some $m, n \in \mathbb{Z}$. Since $\mathrm{FK}_{34}(A) = 0$, $\mathrm{FK}_{24}(A) = 0$ and $\mathrm{FK}_{14}(A) = 0$, the four maps $r_{234}^2 \colon \mathrm{FK}_{234}(A) \to \mathrm{FK}_2(A)$, $r_{234}^3 \colon \mathrm{FK}_{234}(A) \to \mathrm{FK}_3(A)$, $r_{124}^2 \colon \mathrm{FK}_{124}(A) \to \mathrm{FK}_2(A)$ and $r_{134}^3 \colon \mathrm{FK}_{134}(A) \to \mathrm{FK}_3(A)$ are isomorphisms, so $r_{124}^2(g_{124})$ and $r_{234}^2(g_{234}) = mr_{124}^2(g_{124})$ both generate $\mathrm{FK}_2(A)$, and $r_{134}^3(g_{134})$ and $r_{234}^3(g_{234}) = nr_{134}^3(g_{134})$ both generate $\mathrm{FK}_3(A)$, so $m, n \in \{\pm 1\}$. By replacing g_{124} with $-mg_{124}$ and g_{134} with $-ng_{134}$ the required is fulfilled.

In the discussion after the proof of Lemma 5.9 in [MN], we have that the natural homomorphism from $\mathrm{KK}(X_1; R_{12344}, A)$ to $\mathrm{Hom}(\mathrm{FK}(R_{12344}), \mathrm{FK}(A))$ is an isomorphism. Since $\mathrm{FK}(A)$ and $\mathrm{FK}(R_{12344})$ are isomorphic as $\mathcal{NT}$-modules, we have that A and R_{12344} are $\mathrm{KK}(X_1)$-equivalent. □

Lemma 3.2. *There exists an exact triangle*

$$\begin{tikzcd}
R_{1234} \arrow[dr, "\varphi"'] & & R_{12344} \arrow[ll, "\pi"', "\circ" marking] \\
& R_{124} \oplus R_{134} \oplus R_{234} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

satisfying that $\bar{\varphi} = (i_{124}^{1234}, i_{134}^{1234}, i_{234}^{1234}) \in \bigoplus \mathcal{NT}(ij4, 1234)$*, that* $\bar{\pi}$ *generates the group* $\mathcal{NT}(1234, 12344)$*, and that* $\bar{\iota} = (f^{124}, f^{134}, f^{234}) \in \bigoplus \mathcal{NT}(12344, ij4)$ *with each* f^{ij4} *generating* $\mathcal{NT}(12344, ij4)$ *respectively.*

Proof. Let $\varphi \colon R_{1234} \to R_{124} \oplus R_{134} \oplus R_{234}$ be given by restriction to subsets; then $\bar{\varphi} = (i_{124}^{1234}, i_{134}^{1234}, i_{234}^{1234})$. Constructing the mapping cone $\triangle_\varphi$ of φ, we get an exact triangle

$$\begin{tikzcd}
R_{1234} \arrow[dr, "\varphi"'] & & \mathrm{S}\triangle_\varphi \arrow[ll, "\pi"', "\circ" marking] \\
& R_{124} \oplus R_{134} \oplus R_{234} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

and by applying $\mathrm{KK}_*(X_1; R_Y, -) = \mathrm{FK}_Y$ and calculating $\mathrm{FK}_Y(\varphi)$, one sees that $\mathrm{FK}_Y(\mathrm{S}\triangle_\varphi)$ and $\mathrm{FK}_Y(R_{12344})$ are isomorphic for all $Y \in \mathbb{LC}(X_1)^*$, cf. Diagram (3.2).

Furthermore one sees that $\mathrm{FK}_{ij4}(\iota)$ are isomorphisms, and that $\mathrm{FK}_{1234}(\iota)$ is surjective as $\mathrm{FK}_{1234}(\varphi)$ is injective and (using standard generators) is given by $\mathbb{Z} \ni x \mapsto (x, x, x) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Using that $\mathrm{FK}_Y(\iota)$ respects the natural transformations, and that the natural transformation $\mathrm{FK}_{124}(\bigoplus R_{ij4}) \oplus \mathrm{FK}_{134}(\bigoplus R_{ij4}) \to \mathrm{FK}_{1234}(\bigoplus R_{ij4})$ is given by $\mathbb{Z} \oplus \mathbb{Z} \ni (x, y) \mapsto (x, y, 0) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (using standard generators), one can then check that $\mathrm{FK}_{124}(\mathrm{S}\triangle_\varphi) \oplus \mathrm{FK}_{134}(\mathrm{S}\triangle_\varphi) \to \mathrm{FK}_{1234}(\mathrm{S}\triangle_\varphi)$ is an isomorphism. Hence $\mathrm{S}\triangle_\varphi$ and R_{12344} are $\mathrm{KK}(X_1)$-equivalent by Lemma 3.1.

Therefore π and ι induce natural transformations, and since all the involved groups of natural transformations are cyclic and free, we may write $\bar{\pi} = nf_{1234}$ with f_{1234} generating $\mathcal{NT}(1234, 12344)$ and $\bar{\iota} = (n_{ij4} f^{ij4})$ with f^{ij4} generating the group $\mathcal{NT}(12344, ij4)$.

Then $\mathrm{FK}_{ij4}(\iota) = n_{ij4} \, \mathrm{FK}_{ij4}(\hat{f}^{ij4})$, since $\mathrm{FK}_{ij4}(R_{124} \oplus R_{134} \oplus R_{234}) = \mathrm{FK}_{ij4}(R_{ij4})$, so as $\mathrm{FK}_{ij4}(\iota)$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$, we see that $n_{ij4} = \pm 1$.

But $\mathrm{FK}_Y(\pi) = 0$ for all Y. However, since $\mathrm{FK}_{ij4}(R_{1234}) = 0$ and $\mathrm{FK}_{1234}(R_{1234}) = \mathbb{Z}$, we get by applying $\mathrm{KK}_*(X_1; -, R_{1234})$ to the exact triangle that $\bar{\pi} = nf_{1234}$ on R_{1234} is an isomorphism $\mathbb{Z} \to \mathbb{Z}[1]$, hence $n = \pm 1$. □

Lemma 3.3. *There exists an exact triangle*

$$\begin{tikzcd}
R_{12344} \arrow[dr, "\pi"'] & & R_4 \arrow[ll, "\iota", "\circ" description] \\
& R_{14} \oplus R_{24} \oplus R_{34} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

satisfying that $\bar{\varphi} = (i_4^{14}, i_4^{24}, i_4^{34}) \in \bigoplus \mathcal{NT}(4, k4)$*, that* $\bar{\iota}$ *generates* $\mathcal{NT}(12344, 4)$*, and that* $\bar{\pi} = (f_{14}, f_{24}, f_{34}) \in \bigoplus \mathcal{NT}(k4, 12344)$ *with each* f_{k4} *generating the group* $\mathcal{NT}(k4, 12344)$ *respectively.*

Proof. Let $\varphi\colon R_{14} \oplus R_{24} \oplus R_{34} \to M_3(R_4)$ be given by restriction to subsets such that $\bar{\varphi} = (i_4^{14}, i_4^{24}, i_4^{34})$ and construct the mapping cone C_{φ} of φ. By calculating $\mathrm{FK}_Y(\varphi)$ and by applying FK_Y to the mapping cone triangle

$$\begin{tikzcd}
\mathsf{C}_{\varphi} \arrow[dr, "\pi"'] & & R_4 \arrow[ll, "\iota", "\circ" description] \\
& R_{14} \oplus R_{24} \oplus R_{34} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

we see that $\mathrm{FK}_Y(\mathsf{C}_{\varphi}) \cong \mathrm{FK}_Y(R_{12344})$ for all $Y \in \mathbb{LC}(X_1)^*$.

Furthermore we see that $\mathrm{FK}_4(\iota)$ and $\mathrm{FK}_k(\pi)$ are isomorphisms, and that $\mathrm{FK}_{ij4}(\pi)$ and $\mathrm{FK}_{1234}(\pi)$ are injective as $\mathrm{FK}_{ij4}(\varphi)$ and $\mathrm{FK}_{1234}(\varphi)$ are surjective and (by using standard generators) are given by $\mathbb{Z} \oplus \mathbb{Z} \ni (x, y) \mapsto x + y \in \mathbb{Z}$ respectively $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \ni (x, y, z) \mapsto x + y + z \in \mathbb{Z}$.

Using that $\mathrm{FK}_Y(\pi)$ respects the natural transformations, and that the natural transformation $\mathrm{FK}_{124}(\bigoplus R_{k4}) \oplus \mathrm{FK}_{134}(\bigoplus R_{k4}) \to \mathrm{FK}_{1234}(\bigoplus R_{k4})$ is given by $(\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}) \ni (x, y, z, w) \mapsto (x + z, y, w) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (using standard generators), one can then check that $\mathrm{FK}_{124}(\mathsf{C}_{\varphi}) \oplus \mathrm{FK}_{134}(\mathsf{C}_{\varphi}) \to \mathrm{FK}_{1234}(\mathsf{C}_{\varphi})$ is an isomorphism. Hence C_{φ} and R_{12344} are $\mathrm{KK}(X_1)$-equivalent by Lemma 3.1.

Therefore π and ι induce natural transformations, so we may write $\bar{\iota} = nf^4$ with f^4 generating $\mathcal{NT}(12344, 4)$ and $\bar{\pi} = (n_{k4} f_{k4})$ with f_{k4} generating the group $\mathcal{NT}(k4, 12344)$.

As $\mathrm{FK}_4(\iota) = n\, \mathrm{FK}_4(\hat{f}^4)$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}[1]$, we see that $n = \pm 1$. And as $\mathrm{FK}_k(R_{14} \oplus R_{24} \oplus R_{34}) = \mathrm{FK}_k(R_k)$, we see that $\mathrm{FK}_k(\pi) = n_{k4}\, \mathrm{FK}_k(\hat{f}_{k4})$, so as $\mathrm{FK}_k(\pi)$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$, we see that $n_{k4} = \pm 1$. ☐

Lemma 3.4. *There exist natural transformations* $f_{14}, f_{24}, f_{34}, f^{124}, f^{134}, f^{234}$ *such that* $\langle f_{k4} \rangle = \mathcal{NT}(k4, 12344)$ *and* $\langle f^{ij4} \rangle = \mathcal{NT}(12344, ij4)$ *and such that the sequences*

$$\begin{tikzcd}
\mathrm{FK}_{1234}(A) \arrow[rr, "f_{m4} i_4^{m4} \delta_n^4 r_{1234}^n", "\circ" description] & & \mathrm{FK}_{12344}(A) \arrow[dl, "(f^{ij4})"] \\
& \mathrm{FK}_{124}(A) \oplus \mathrm{FK}_{134}(A) \oplus \mathrm{FK}_{234}(A) \arrow[ul, "(i_{ij4}^{1234})"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{FK}_{12344}(A) \arrow[rr, "\delta_m^4 r_{mn4}^m f^{mn4}", "\circ" description] & & \mathrm{FK}_4(A) \arrow[dl, "(i_4^{k4})"] \\
& \mathrm{FK}_{14}(A) \oplus \mathrm{FK}_{24}(A) \oplus \mathrm{FK}_{34}(A) \arrow[ul, "(f_{k4})"] &
\end{tikzcd}$$

are exact for all C^*-algebras A over X_1 and all $m, n \in \{1, 2, 3\}$ with $m \neq n$.

Proof. This follows from Lemmas 3.2 and 3.3 by applying $\mathrm{KK}_*(X_1; -, A)$ and using that by the Diagram (3.1) the transformation $f_{m4} i_4^{m4} \delta_n^4 r_{1234}^n$ generates the group $\mathcal{NT}(1234, 12344)$ and $\delta_m^4 r_{mn4}^m f^{mn4}$ generates $\mathcal{NT}(12344, 4)$. □

3.2. **A classification result.**

Proposition 3.5. *Let A and B be C^*-algebras over X_1 and assume that the maps $\delta_m^4 \colon \mathrm{FK}_m^n(A) \to \mathrm{FK}_4^{1-n}(A)$ and $\delta_m^4 \colon \mathrm{FK}_m^n(B) \to \mathrm{FK}_4^{1-n}(B)$ vanish for some $m \in \{1, 2, 3\}$ and some $n \in \{0, 1\}$. Then any homomorphism $\varphi \colon \mathrm{FK}(A) \to \mathrm{FK}(B)$ can be uniquely extended to a homomorphism $\varphi' \colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. Furthermore, if φ is an isomorphism then so is φ'.*

Proof. Let $\varphi \colon \mathrm{FK}(A) \to \mathrm{FK}(B)$ be a homomorphism. We may extend it by defining $\varphi_{12344} \colon \mathrm{FK}_{12344}(A) \to \mathrm{FK}_{12344}(B)$ by the following diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{FK}_{12344}^{1-n}(A) \arrow[r, "(f^{ij4})"] \arrow[d, dotted, "\varphi_{12344}^{1-n}"] & \mathrm{FK}_{124}^{1-n}(A) \oplus \mathrm{FK}_{134}^{1-n}(A) \oplus \mathrm{FK}_{234}^{1-n}(A) \arrow[r, "(i_{ij4}^{1234})"] \arrow[d, "\varphi_{124}^{1-n} \oplus \varphi_{134}^{1-n} \oplus \varphi_{234}^{1-n}"] & \mathrm{FK}_{1234}^{1-n}(A) \arrow[d, "\varphi_{1234}^{1-n}"] \\
0 \arrow[r] & \mathrm{FK}_{12344}^{1-n}(B) \arrow[r, "(f^{ij4})"] & \mathrm{FK}_{124}^{1-n}(B) \oplus \mathrm{FK}_{134}^{1-n}(B) \oplus \mathrm{FK}_{234}^{1-n}(B) \arrow[r, "(i_{ij4}^{1234})"] & \mathrm{FK}_{1234}^{1-n}(B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{FK}_4^n(A) \arrow[r, "(i_4^{k4})"] \arrow[d, "\varphi_4^n"] & \mathrm{FK}_{14}^n(A) \oplus \mathrm{FK}_{24}^n(A) \oplus \mathrm{FK}_{34}^n(A) \arrow[r, "(f_{k4})"] \arrow[d, "\varphi_{14}^n \oplus \varphi_{24}^n \oplus \varphi_{34}^n"] & \mathrm{FK}_{12344}^n(A) \arrow[r] \arrow[d, dotted, "\varphi_{12344}^n"] & 0 \\
\mathrm{FK}_4^n(B) \arrow[r, "(i_4^{k4})"] & \mathrm{FK}_{14}^n(B) \oplus \mathrm{FK}_{24}^n(B) \oplus \mathrm{FK}_{34}^n(B) \arrow[r, "(f_{k4})"] & \mathrm{FK}_{12344}^n(B) \arrow[r] & 0
\end{tikzcd}$$

By Lemma 3.4 the four horizontal sequences in the diagrams are exact, hence φ_{12344} is well-defined and is bijective if φ is an isomorphism.

By construction φ_{12344}^n respects the natural transformations f_{k4} and φ_{12344}^{1-n} respects the naturals transformations f^{ij4}. Since (f^{ij4}) is injective on $\mathrm{FK}_{12344}^{1-n}(B)$ and since (f_{k4}) is surjective on $\mathrm{FK}_{12344}^n(A)$, it suffices to check that

$$(f^{ij4})\varphi_{12344}^{1-n} f_{k4} = (f^{ij4}) f_{k4} \varphi_{k4}^{1-n} \quad \text{and} \quad \varphi_{ij4}^n f^{ij4}(f_{k4}) = f^{ij4} \varphi_{12344}^n (f_{k4}).$$

And this holds by construction of φ_{12344} as

$$f^{ij4} f_{k4} \varphi_{k4} = \varphi_{ij4} f^{ij4} f_{k4}$$

since $f^{ij4} f_{k4} \in \mathcal{NT}(k4, ij4)$.

Since the natural transformations in FK' are generated by the natural transformations in FK together with the natural transformations f_{k4} and f^{ij4}, we see that the extended φ respects all the natural transformations in FK', hence it is an $\mathcal{NT}'$-morphism between $\mathrm{FK}'(A)$ and $\mathrm{FK}'(B)$. □

Observation 3.6. A tight, purely infinite, nuclear, separable C^*-algebra A over a finite T_0-space X is of real rank zero if and only if the boundary map $\delta_{Y \setminus U}^U$ vanishes on $\mathrm{K}_0(A(Y \setminus U))$ for all $Y \in \mathbb{LC}(X)$ and all $U \in \mathcal{O}(Y)$. This follows from the fact that all Kirchberg algebras have real rank zero combined with the following result of L. G. Brown and G. K. Pedersen, cf. [BP91, 3.14]: Given an extension $I \hookrightarrow B \twoheadrightarrow B/I$ of C^*-algebras, B has real rank zero if and only if I and B/I have real rank zero and projections in B/I lift to projections in B.

Corollary 3.7. *Let A and B be C^*-algebras in the bootstrap class over X_1 and with A of real rank zero. Then any isomorphism between $\mathrm{FK}(A)$ and $\mathrm{FK}(B)$ lifts to a $\mathrm{KK}(X_1)$-equivalence.*

Proof. Since A is of real rank zero, $\delta_2^4\colon \mathrm{FK}_2^0(A) \to \mathrm{FK}_4^1(A)$ vanishes by [BP91, 3.14], and since $\mathrm{FK}(A)$ and $\mathrm{FK}(B)$ are isomorphic, $\delta_2^4\colon \mathrm{FK}_2^0(B) \to \mathrm{FK}_4^1(B)$ also vanishes. By Proposition 3.5 the isomorphism therefore lifts to an isomorphism between $\mathrm{FK}'(A)$ and $\mathrm{FK}'(B)$, and by [MN, 5.14] this lifts to a $\mathrm{KK}(X_1)$-equivalence. □

Definition 3.8. Let A and B be unital C^*-algebras over X. Then $\varphi : \mathrm{FK}(A) \to \mathrm{FK}(B)$ is a *homomorphism that preserves the unit* if φ is a homomorphism of $\mathcal{NT}$-modules and $\varphi_X([1_A]) = [1_B]$ in $\mathrm{FK}_X(A) = \mathrm{FK}_X(B)$. We say that φ is an *isomorphism that preserves the unit* if φ is an isomorphism of $\mathcal{NT}$-modules that preserves the unit.

Combining this with [Kir00, 4.3] and [RR07, 2.1,3.2], we obtain the following corollary.

Corollary 3.9. *Let A and B be purely infinite, nuclear, separable C^*-algebras that are tight over X_1 and whose simple subquotients lie in the bootstrap class. Assume that A has real rank zero.*

1. *If A and B are stable, then every isomorphism from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ can be lifted to a $*$-isomorphism from A to B.*
2. *If A and B are unital, then every isomorphism from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ that preserves the unit can be lifted to a $*$-isomorphism from A to B.*

Remark 3.10. The space $X_4 = X_1^{\mathrm{op}}$ has been studied in [BK] where it is shown that the indecomposable transformations for X_1^{op} are

$$\begin{tikzcd}
 & 234 \arrow[r, "r"] \arrow[dr, "r"'] & 34 \arrow[dr, "r"] & & 1 \arrow[dr, "i"] & \\
1234 \arrow[ur, "r"] \arrow[r, "r"] \arrow[dr, "r"'] & 134 \arrow[ur, "r"] \arrow[dr, "r"'] & 24 \arrow[r, "r"] & 4 \arrow[ur, "\delta", "\circ" marking] \arrow[r, "\delta", "\circ" marking] \arrow[dr, "\delta", "\circ" marking] & 2 \arrow[r, "i"] & 1234 \\
 & 124 \arrow[ur, "r"] \arrow[r, "r"] & 14 \arrow[ur, "r"] & & 3 \arrow[ur, "i"] &
\end{tikzcd}.$$

It is straightforward to copy the methods of Meyer and Nest in [MN] to construct a refined filtrated K-theory for which there is a UCT; for X_1^{op} the extra representing object is the mapping cone of a generator of $\mathcal{NT}(14, 234)$. The methods we used for the spaces X_1 apply to X_1^{op} as well since the boundary maps δ are placed in similar places in the structure diagrams for $\mathcal{NT}$ of X_1^{op}.

4. ANOTHER COUNTEREXAMPLE

Consider the space $X_2 = \{1, 2, 3, 4\}$ with $\mathcal{O}(X_2) = \{\emptyset, 4, 34, 234, 134, X_2\}$. Then $1 < 3$, $2 < 3$ and $3 < 4$, $\mathbb{LC}(X_2)^* = \{4, 34, 234, 134, 1234, 3, 23, 13, 123, 1, 2\}$, and its Hasse diagram is

$$\begin{tikzcd}
1 & & 2 \\
& 3 \arrow[ul] \arrow[ur] & \\
& 4 \arrow[u] &
\end{tikzcd} .$$

The indecomposable transformations in the category $\mathcal{NT}$ have been studied in detail in [Ben10, 6.1.2] and are the maps in the following diagram:

$$\begin{tikzcd}
& 134 \arrow[r, "r"] \arrow[dr, "i"] & 13 \arrow[dr, "i"] & & 1 \arrow[dr, "\circ" marking, "\delta"] & \\
34 \arrow[ur, "i"] \arrow[r, "r"] \arrow[dr, "i"] & 3 \arrow[ur, "i"] \arrow[dr, "i"] & 1234 \arrow[r, "r"] & 123 \arrow[ur, "r"] \arrow[r, "\circ" marking, "\delta"] \arrow[dr, "r"] & 4 \arrow[r, "i"] & 34 \\
& 234 \arrow[r, "r"] \arrow[ur, "i"] & 23 \arrow[ur, "i"] & & 2 \arrow[ur, "\circ" marking, "\delta"] &
\end{tikzcd}$$

As with the first counterexample, there exists a refinement FK' of FK for which there is a UCT, cf. [Ben10, 6.1], so for A and B in the bootstrap class $\mathcal{B}(X_2)$ one can lift an isomorphism between $\mathrm{FK}'(A)$ and $\mathrm{FK}'(B)$ to a $\mathrm{KK}(X_2)$-equivalence.

For X_2 one constructs an extra representing object R_{12334} as the mapping cone of a generator of the cyclic free group $\mathcal{NT}(23, 134)$, and its filtrated K-theory is then

$$\begin{tikzcd}
& 0 \arrow[r, "r"] \arrow[dr, "i"] & \mathbb{Z} \arrow[dr, "i"] & & \mathbb{Z} \arrow[dr, "\circ" marking, "\delta"] & \\
\mathbb{Z}[1] \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "i"] & 0 \arrow[ur, "i"] \arrow[dr, "i"] & \mathbb{Z} \arrow[r, "r"] & \mathbb{Z}^2 \arrow[ur, "r"] \arrow[r, "\circ" marking, "\delta"] \arrow[dr, "r"] & \mathbb{Z}[1] \arrow[r, "i"] & \mathbb{Z}[1] \\
& 0 \arrow[r, "r"] \arrow[ur, "i"] & \mathbb{Z} \arrow[ur, "i"] & & \mathbb{Z} \arrow[ur, "\circ" marking, "\delta"] &
\end{tikzcd}$$

where the three maps i_{13}^{123}, r_{1234}^{123} and i_{23}^{123} are given by the three coordinate embeddings $\mathbb{Z} \to \mathbb{Z}^3/(1,1,1)$, the three maps r_{123}^{1}, δ_{123}^{4} and r_{123}^{2} are given by the three projections $\mathbb{Z}^3/(1,1,1) \to \mathbb{Z}^2/(1,1)$ onto coordinate hyperplanes, and the three maps δ_1^{34}, i_4^{34} and δ_2^{34} are the identity.

Since $\mathrm{pd}(\mathrm{FK}(R_{12334})) = 1$, we see that for any C^*-algebra A over X_2 that lies in the bootstrap class over X_2, A and R_{12334} will be $\mathrm{KK}(X_2)$-equivalent if and only if the groups $\mathrm{FK}_Y(A)$ and $\mathrm{FK}_Y(R_{12334})$ are isomorphic for all $Y \in \mathbb{LC}(X_2)^*$ and the natural transformation $\mathrm{FK}_{13}(A) \oplus \mathrm{FK}_{1234}(A) \to \mathrm{FK}_{123}(A)$ is an isomorphism, cf. Lemma 3.1. The indecomposable transformations in the ring $\mathcal{NT}'$ fit into the following diagram:

$$\begin{tikzcd}
& 134 \arrow[dr] & & 13 \arrow[dr, "i"] & & 1 \arrow[dr, "\circ" marking, "\delta"] & \\
34 \arrow[ur, "i"] \arrow[r, "r"] \arrow[dr, "i"] & 3 \arrow[r] & 12334 \arrow[ur] \arrow[r] \arrow[dr] & 1234 \arrow[r, "r"] & 123 \arrow[ur, "r"] \arrow[r, "\circ" marking, "\delta"] \arrow[dr, "r"] & 4 \arrow[r, "i"] & 34 \\
& 234 \arrow[ur] & & 23 \arrow[ur, "i"] & & 2 \arrow[ur, "\circ" marking, "\delta"] &
\end{tikzcd} \tag{4.1}$$

4.1. **The refined invariant.**

Lemma 4.1. *There exists an exact triangle*

$$\begin{tikzcd}
R_{123} \arrow[dr, "\varphi"'] & & R_{12334} \arrow[ll, "\pi"', "\circ" marking] \\
& R_{13} \oplus R_{1234} \oplus R_{23} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

satisfying that $\bar{\varphi} = (i_{13}^{123}, r_{1234}^{123}, i_{23}^{123})$*, that* $\bar{\pi}$ *generates the group* $\mathcal{NT}(123, 12334)$*, and that* $\bar{\iota} = (f^{13}, f^{1234}, f^{23})$ *with* f^Y *generating* $\mathcal{NT}(12344, Y)$*.*

Proof. Let $\varphi\colon R_{123} \to R_{13} \oplus R_{1234} \oplus R_{23}$ be given by inclusion respectively restrictions to subspaces, such that $\bar{\varphi} = (i_{13}^{123}, r_{1234}^{123}, i_{23}^{123})$. The proof is similar to the proof of Lemma 3.2. Here $\mathrm{FK}(R_{123})$ is used to establish that $\bar{\pi}$ is a generator, and FK_Y is used for f^Y. ☐

Lemma 4.2. *There exists an exact triangle*

$$\begin{tikzcd}
R_{12334} \arrow[dr, "\pi"'] & & R_{34} \arrow[ll, "\iota"', "\circ" marking] \\
& R_{134} \oplus R_3 \oplus R_{234} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

satisfying that $\bar{\varphi} = (i_{34}^{134}, r_{34}^{3}, i_{34}^{234})$*, that* $\bar{\iota}$ *generates* $\mathcal{NT}(12334, 34)$*, and that* $\bar{\pi} = (f_{134}, f_3, f_{234})$ *with each* f_Y *generating the group* $\mathcal{NT}(Y, 12344)$ *respectively.*

Proof. Let $\varphi\colon R_{134} \oplus R_3 \oplus R_{234} \to M_3(R_{34})$ be given by inclusions respectively restriction to a subspace, such that $\bar{\varphi} = (i_{34}^{134}, r_{34}^{3}, i_{34}^{234})$. The proof is similar to the proof of Lemma 3.3. Here FK_4 is used to establish that $\bar{\iota}$ is a generator, and FK_Y is used for f_Y. ☐

Lemma 4.3. *There exist natural transformations* $f_{134}, f_3, f_{234}, f^{13}, f^{1234}, f^{23}$ *such that* $\langle f_Y \rangle = \mathcal{NT}(Y, 12334)$ *and* $\langle f^Y \rangle = \mathcal{NT}(12334, Y)$ *and such that the sequences*

$$\begin{tikzcd}
\mathrm{FK}_{123}(A) \arrow[rr, "f_{134} i_4^{134} \delta_{123}^4", "\circ" marking] & & \mathrm{FK}_{12334}(A) \arrow[dl, "{(f^{13}, f^{1234}, f^{23})}"] \\
& \mathrm{FK}_{13}(A) \oplus \mathrm{FK}_{1234}(A) \oplus \mathrm{FK}_{23}(A) \arrow[ul, "{(i_{13}^{123}, r_{1234}^{123}, i_{23}^{123})}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{FK}_{12334}(A) \arrow[rr, "r_4^{34} \delta_{123}^4 i_{23}^{123} f^{23}", "\circ" marking] & & \mathrm{FK}_{34}(A) \arrow[dl, "{(i_{34}^{134}, r_{34}^{3}, i_{34}^{234})}"] \\
& \mathrm{FK}_{134}(A) \oplus \mathrm{FK}_3(A) \oplus \mathrm{FK}_{234}(A) \arrow[ul, "{(f_{134}, f_3, f_{234})}"] &
\end{tikzcd}$$

are exact for all C^**-algebras* A *over* X_2*.*

Proof. This follows from Lemmas 4.1 and 4.2 by applying $\mathrm{KK}_*(X_2; -, A)$ and using that by the Diagram (4.1) the transformation $f_{134} i_4^{134} \delta_{123}^4$generates $\mathcal{NT}(123, 12334)$ and the transformation $r_4^{34} \delta_{123}^4 i_{23}^{123} f^{23}$generates $\mathcal{NT}(12334, 34)$. ☐

4.2. **A classification result.** A slightly more general result, like the result in Section 3.2, can be obtained, but we state a weaker result to ease notation.

Proposition 4.4. *Let A and B be C^*-algebras over X_2 and assume that A and B have real rank zero. Then any homomorphism $\varphi\colon \mathrm{FK}(A) \to \mathrm{FK}(B)$ can be uniquely extended to a homomorphism $\varphi'\colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. Furthermore, if φ is an isomorphism then so is φ'.*

Proof. The proof is similar to the proof of Theorem 3.5. Since A and B have real rank zero, $\delta^4_{123}\colon \mathrm{FK}^0_{123}(A) \to \mathrm{FK}^1_4(A)$ and $\delta^4_{123}\colon \mathrm{FK}^0_{123}(B) \to \mathrm{FK}^1_4(B)$ vanish, so by Lemma 4.3 the horizontal sequences in the following diagram are exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{FK}^1_{12334}(A) \arrow[r] \arrow[d, dotted, "\varphi^1_{12334}"] & \mathrm{FK}^1_{13}(A) \oplus \mathrm{FK}^1_{1234}(A) \oplus \mathrm{FK}^1_{23}(A) \arrow[r] \arrow[d, "\varphi^1_{13}\oplus\varphi^1_{1234}\oplus\varphi^1_{23}"] & \mathrm{FK}^1_{123}(A) \arrow[d, "\varphi^1_{123}"] \\
0 \arrow[r] & \mathrm{FK}^1_{12334}(B) \arrow[r] & \mathrm{FK}^1_{13}(B) \oplus \mathrm{FK}^1_{1234}(B) \oplus \mathrm{FK}^1_{23}(B) \arrow[r] & \mathrm{FK}^1_{123}(B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{FK}^0_{4}(A) \arrow[r] \arrow[d, "\varphi^0_4"] & \mathrm{FK}^0_{134}(A) \oplus \mathrm{FK}^0_{3}(A) \oplus \mathrm{FK}^0_{234}(A) \arrow[r] \arrow[d, "\varphi^0_{134}\oplus\varphi^0_3\oplus\varphi^0_{234}"] & \mathrm{FK}^0_{12334}(A) \arrow[r] \arrow[d, dotted, "\varphi^0_{12334}"] & 0 \\
\mathrm{FK}^0_{4}(B) \arrow[r] & \mathrm{FK}^0_{134}(B) \oplus \mathrm{FK}^0_{3}(B) \oplus \mathrm{FK}^0_{234}(B) \arrow[r] & \mathrm{FK}^0_{12334}(B) \arrow[r] & 0
\end{tikzcd}$$

so we may recover FK^1_{12334} as the kernel of $(i^{123}_{13}, r^{123}_{1234}, i^{123}_{13})$ and FK^0_{12334} as the cokernel of $(i^{134}_{34}, r^3_{34}, i^{234}_{34})$, as in the proof of Theorem 3.5. □

Corollary 4.5. *Let A and B be C^*-algebras in the bootstrap class over X_2 and assume that A has real rank zero. Then any isomorphism between $\mathrm{FK}(A)$ and $\mathrm{FK}(B)$ lifts to a $\mathrm{KK}(X_2)$-equivalence.*

Proof. Since $\mathrm{FK}(A)$ and $\mathrm{FK}(B)$ are isomorphic, $\delta^4_{123}\colon \mathrm{FK}^0_{123}(B) \to \mathrm{FK}^1_4(B)$ vanishes, so the proof of Proposition 4.4 applies, hence the isomorphism lifts to an isomorphism between $\mathrm{FK}'(A)$ and $\mathrm{FK}'(B)$ and by [Ben10, 6.1.22] this lifts to a $\mathrm{KK}(X_2)$-equivalence. □

Corollary 4.6. *Let A and B be purely infinite, nuclear, separable C^*-algebras that are tight over X_2 and whose simple subquotients lie in the bootstrap class. Assume that A has real rank zero.*

1. *If A and B are stable, then every isomorphism from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ can be lifted to a $*$-isomorphism from A to B.*
2. *If A and B are unital, then every isomorphism from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ that preserves the unit can be lifted to a $*$-isomorphism from A to B.*

Remark 4.7. The space $X_5 = X_2^{\text{op}}$ has been studied in [BK] where it is shown that the indecomposable transformations for X_2^{op} are

$$\begin{tikzcd}
 & 23 \arrow[r, "i"] \arrow[dr, "r"] & 234 \arrow[dr, "r"] & & 1 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "r"] & 1234 \arrow[ur, "r"] \arrow[dr, "r"] & 3 \arrow[r, "i"] & 34 \arrow[ur, "\delta", "\circ" marking] \arrow[r, "r"] \arrow[dr, "\delta", "\circ" marking] & 4 \arrow[r, "\delta", "\circ" marking] & 123 \\
 & 13 \arrow[ur, "r"] \arrow[r, "i"] & 134 \arrow[ur, "r"] & & 2 \arrow[ur, "i"] &
\end{tikzcd}.$$

As with X_1^{op}, cf. Remark 3.10, it is straightforward to copy the methods of Meyer and Nest in [MN] to construct a refined filtrated K-theory for which there is a UCT; for X_2^{op} the extra representing object is the mapping cone of a generator of $\mathcal{NT}(134, 23)$. And as with X_1^{op}, the methods we used for the spaces X_1 and X_2 apply to X_2^{op} since the boundary maps δ are placed in similar places in the structure diagrams for $\mathcal{NT}$ of X_2^{op}.

5. A THIRD COUNTEREXAMPLE

Consider the space $X_3 = \{1, 2, 3, 4\}$ with $\mathcal{O}(X_3) = \{\emptyset, 4, 24, 34, 234, X_3\}$. Then $1 < 2$, $1 < 3$, $2 < 4$, $3 < 4$, $\mathbb{LC}(X_3)^* = \{4, 24, 34, 234, 1234, 123, 12, 13, 1, 2, 3\}$ and its Hasse diagram is

$$\begin{tikzcd}
 & 1 & \\
2 \arrow[ur] & & 3 \arrow[ul] \\
 & 4 \arrow[ul] \arrow[ur] &
\end{tikzcd}.$$

The indecomposable transformations in the category $\mathcal{NT}$ have been studied in detail in [Ben10, 6.2.2] and are displayed in the following diagram:

$$\begin{tikzcd}
 & 12 \arrow[r, "\delta", "\circ" marking] \arrow[dr, "i"] & 34 \arrow[dr, "i"] & & 3 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "\delta", "\circ" marking] \arrow[dr, "r"] & 4 \arrow[ur, "r"] \arrow[dr, "r"] & 1 \arrow[r, "\delta", "\circ" marking] & 234 \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "r"] & 1234 \arrow[r, "r"] & 123 \\
 & 13 \arrow[ur, "i"] \arrow[r, "\delta", "\circ" marking] & 24 \arrow[ur, "i"] & & 2 \arrow[ur, "i"] &
\end{tikzcd}$$

The methods used for the spaces X_1 and X_2 do not apply to X_3 since the boundary maps δ are placed radically differently in the structure diagram for $\mathcal{NT}$ of X_3. In fact, for this space X_3 there does exist tight, nuclear, separable, purely infinite C^*-algebras A and B over X_3 of real rank zero that are not $\text{KK}(X_3)$-equivalent but have isomorphic filtrated K-theory.

Proof of Theorem 1.2. The construction is similar to the one of R. Meyer and R. Nest in [MN, p. 27ff] and some of the details are carried out in [Ben10, 6.2.4]. Put $P_Y(Z) = \mathcal{NT}(Y, Z)$. Consider the injective map $j\colon P_{234} \to P_{24} \oplus P_1[1] \oplus P_{34}$ induced by three generators of the groups $\mathcal{NT}(24, 234)$, $\mathcal{NT}(1, 234)$ and $\mathcal{NT}(34, 234)$, and let M denote the cokernel. Let $k \geq 2$ and put $M_k = M \otimes \mathbb{Z}/k$. Then M_k is

$$\begin{tikzcd}
& 0 \arrow[r, "\circ" marking] \arrow[dr] & \mathbb{Z}/k \arrow[dr] & & \mathbb{Z}/k \arrow[dr] & \\
\mathbb{Z}/k \arrow[ur] \arrow[r, "\circ" marking] \arrow[dr] & 0 \arrow[ur] \arrow[dr] & \mathbb{Z}/k[1] \arrow[r, "\circ" marking] & (\mathbb{Z}/k)^2 \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z}/k \arrow[r] & \mathbb{Z}/k \\
& 0 \arrow[ur] \arrow[r, "\circ" marking] & \mathbb{Z}/k \arrow[ur] & & \mathbb{Z}/k \arrow[ur] &
\end{tikzcd}$$

and has projective dimension 2, and

$$0 \longrightarrow P_{234} \longrightarrow P_{234} \oplus P_{24} \oplus P_1[1] \oplus P_{34} \longrightarrow P_{24} \oplus P_1[1] \oplus P_{34} \longrightarrow M_k \longrightarrow 0$$

is a projective resolution of M_k. Notice that the boundary maps from even to odd parts in M_k vanish. There exists in the bootstrap class over X_3 a C^*-algebra A_k with $\mathrm{FK}(A_k) = M_k$, see [Ben10, 6.2.4] for details. Let

$$Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow A_k$$

be a $\ker \mathrm{FK}$-projective resolution which is a lift of the above projective resolution of M_k, and let

$$\begin{tikzcd}
A_k \arrow[r, equal] & N_0 \arrow[rr] & & N_1 \arrow[rr] \arrow[dl, "\circ" marking] & & N_2 \arrow[rr] \arrow[dl, "\circ" marking] & & N_3 \arrow[rr, equal] \arrow[dl, "\circ" marking] & & N_3 \arrow[rr, equal] \arrow[dl, "\circ" marking] & & \cdots \\
& & P_0 \arrow[ul] & & P_1 \arrow[ll] \arrow[ul] & & P_2 \arrow[ll] \arrow[ul] & & 0 \arrow[ll] \arrow[ul] & & \cdots \arrow[ll] &
\end{tikzcd}$$

be its phantom tower. Then $N_2 \cong_{\mathrm{KK}(X_3)} Q_2$ and the composite map $A_k \to N_2$ lies in $(\ker \mathrm{FK})^2$. Construct B as the mapping cone of $A_k \to N_2$. Then B and $A_k \oplus SN_2$ are not $\mathrm{KK}(X_3)$-equivalent but have $\mathrm{FK}(B) \cong \mathrm{FK}(A_k) \oplus \mathrm{FK}(N_2)[1] = M_k \oplus P_{234}[1]$. See [MN, 4.10, 5.5] for more details.

Since all $\mathrm{KK}(X_3)$-equivalence classes in the bootstrap class over X_3 can be represented by tight, stable, purely infinite, nuclear, separable C^*-algebras over X_3, cf. [MN, 4.6], we can find such C and D with $C \cong_{\mathrm{KK}(X_3)} B$, $D \cong_{\mathrm{KK}(X_3)} A_k \oplus SN_2$ and $\mathrm{FK}(C) \cong \mathrm{FK}(D) \cong \mathrm{FK}(B)$. Since $P_{234}[1]$ is

$$\begin{tikzcd}
& \mathbb{Z}[1] \arrow[r, "\circ" marking] \arrow[dr] & 0 \arrow[dr] & & \mathbb{Z}[1] \arrow[dr] & \\
\mathbb{Z}[1]^2 \arrow[ur] \arrow[r, "\circ" marking] \arrow[dr] & \mathbb{Z} \arrow[ur] \arrow[dr] & 0 \arrow[r, "\circ" marking] & \mathbb{Z}[1] \arrow[ur] \arrow[r] \arrow[dr] & \mathbb{Z}[1] \arrow[r] & \mathbb{Z}[1]^2 \\
& \mathbb{Z}[1] \arrow[ur] \arrow[r, "\circ" marking] & 0 \arrow[ur] & & \mathbb{Z}[1] \arrow[ur] &
\end{tikzcd},$$

we see that the boundary maps from even to odd parts in $\mathrm{FK}(B)$ vanish, so C and D will be of real rank zero as their simple subquotients are Kirchberg algebras and therefore of real rank zero, cf. Observation 3.6. □

Remark 5.1. The real rank zero counter-examples for the space X_3 have torsion in both even and odd degrees. In [ABK], it is shown that for real rank zero C^*-algebras over X_3 with free K_1-groups, isomorphisms on a reduced filtrated K-theory lift to $\mathrm{KK}(X_3)$-equivalences. This reduced filtrated K-theory is defined in [ABK] by disregarding parts of the information in filtrated K-theory, and it is equivalent to the reduced filtered K-theory defined by the second named author in [Res06, 4.1]. It is unknown whether isomorphisms on FK lift to $\mathrm{KK}(X_3)$-equivalences under these conditions.

Remark 5.2. The space X_6 has been studied in [Ben10] where R. Bentmann fails to construct a finite refinement of filtrated K-theory over X_6 that admits a UCT and remarks that it seems unlikely that such a finite refinement exists. So our method cannot be applied for the space X_6. In [Ben10], R. Bentmann constructs tight, stable, purely infinite, nuclear, separable C^*-algebras A and B over X_6 that have isomorphic filtrated K-theory and are not $\mathrm{KK}(X_6)$-equivalent. One can check that the boundary map $\mathrm{FK}_1(A) \to \mathrm{FK}_3(A)$ does not vanish in either degrees, so neither A and B nor the suspensions $\mathrm{S}\,A$ and $\mathrm{S}\,B$ have real rank zero. So there is so far no known real rank zero counter-example for X_6.

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REDUCTION OF FILTERED K-THEORY AND A CHARACTERIZATION OF CUNTZ-KRIEGER ALGEBRAS

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Abstract. For real rank zero C^*-algebras over finite T_0-spaces in a certain class we show that (concrete) filtered K-theory can be recovered from a simplified invariant. This class of spaces contains all spaces X for which filtered K-theory is known to classify Kirchberg X-algebras of real rank zero with simple subquotients in the bootstrap class.

We define another reduced version of filtered K-theory and determine the range on the category of graph C^*-algebras over an arbitrary finite T_0-space X. For real rank zero C^*-algebras over a space in our class whose subquotients have free K_1-groups we show that (concrete) filtered K-theory can be recovered from this reduced invariant.

If X has the property that the reduced invariant classifies Kirchberg X-algebras of real rank zero with simple subquotients in the bootstrap class, then we obtain a characterisation of when an extension of stabilized Cuntz-Krieger algebras is stably isomorphic to a Cuntz-Krieger algebra in terms of a condition on the corresponding six-term exact sequence in K-theory.

1. Introduction

By a seminal result of Eberhard Kirchberg, $\mathrm{KK}(X)$-equivalences between Kirchberg X-algebras, that is, tight, stable, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebras over a space X, lift to X-equivariant *-isomorphisms. With the aim of computing the equivariant bivariant theory $\mathrm{KK}(X)$, Ralf Meyer and Ryszard Nest established in [10] a Universal Coefficient Theorem for filtered K-theory over any finite totally ordered space X. As a result, for such spaces X isomorphisms on filtered K-theory between Kirchberg X-algebras with simple subquotients in the bootstrap class lift to X-equivariant *-isomorphisms. This result was generalised in [2] by the second named author to the case of so-called accordion space defined in Section 2. Building on these results, Søren Eilers, Gunnar Restorff and Efren Ruiz classified in [9] certain classes of real-rank-zero (not necessarily purely infinite) graph algebras using *ordered* filtered K-theory.

On the other hand, Meyer-Nest and the second-named author constructed counterexamples to classification for all six four-point non-accordion spaces. More precisely, for each such X they find two non-$\mathrm{KK}(X)$-equivalent Kirchberg X-algebras with simple subquotients in the bootstrap class whose filtered K-theories are isomorphic (see [2, 10]).

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Despite this obstruction, it had previously been shown by Gunnar Restorff in [13] that filtered K-theory FK, and in fact the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$, is a complete invariant for a certain class of unital, purely infinite, nuclear, separable C^*-algebras with arbitrary finite ideal lattices, namely the Cuntz-Krieger algebras satisfying property (II). Any finite T_0-space can be realized as the primitive ideal space of a Cuntz-Krieger algebra with property (II). Unfortunately, Restorff's result only gives an *internal* classification of Cuntz-Krieger algebras and admits no conclusion concerning when a given Cuntz-Krieger algebra is stably isomorphic to a given purely infinite, nuclear, separable C^*-algebra with the same filtered K-theory.

The Cuntz-Krieger algebras satisfying property (II) have real rank zero. In [1], Gunnar Restorff, Efren Ruiz and the first-named author noted that for five of the six problematic four-point spaces the constructed counterexamples to classification do *not* have real rank zero. They went on to show that for four of these spaces X filtered K-theory is in fact a complete invariant for Kirchberg X-algebras of real rank zero with simple subquotients in the bootstrap class. The four-point non-accordion space for which the constructed counterexample has real rank zero will be denoted by $\mathcal{D}$.

For every Cuntz-Krieger algebra satisfying property (II) the K_1-group of every subquotient is free. The same is true, more generally, for graph algebras with real rank zero. We observe that, for real-rank-zero C^*-algebras over $\mathcal{D}$ satisfying this condition on their K-theory, isomorphisms on the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ lift to $\mathrm{KK}(\mathcal{D})$-equivalences (see Remark 8.15). There are therefore no known counterexamples to classification by filtered K-theory of Kirchberg X-algebras with simple subquotients in the bootstrap class that have the K-theory of a real-rank-zero graph algebra.

1.1. Organization of the paper.

The main focus of this paper is not completeness of filtered K-theory, but reduction of filtered K-theory, and the range of filtered K-theory for graph algebras. The main results are recaptured in Theorem 10.1.

In Section 6, filtered K-theory restricted to a canonical base $\mathrm{FK}_{\mathcal{B}}$ is defined for spaces with a specified boundary decomposition property, and it is shown that the concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}(A)$ of a real rank zero C^*-algebra A is completely determined by the filtered K-theory restricted to a canonical base $\mathrm{FK}_{\mathcal{B}}(A)$.

In Section 7, reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ is defined, and it is shown for spaces with the so-called boundary decomposition property that the concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}(A)$ of a real rank zero C^*-algebra A satisfying that all subquotients have free K_1-groups can be recovered from the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}(A)$. This is of particular interest since in Section 9 we determine the range of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ for graph algebras.

In Section 9, we combine the range result of $\mathrm{FK}_{\mathcal{R}}$ with completeness of $\mathrm{FK}_{\mathcal{R}}$ for some spaces with the boundary decomposition property to determine exactly when an extension of stabilized Cuntz-Krieger algebras is a stabilized Cuntz-Krieger algebra.

2. NOTATION

We follow the notation and definition for graph algebras of Iain Raeburn, cf. [12]. All graphs are assumed to be countable and to satisfy Condition (K), hence all considered graph algebras are separable and of real rank zero. In this article, matrices act from the right and the composite of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is denoted by fg.

Let X be a finite T_0-space. For a subset Y of X, we let $\overline{Y}$ denote the closure of Y, and let $\overline{\partial}Y$ denote the (closed) boundary $\overline{Y} \setminus Y$ of Y. Since X is a finite space, there exists a smallest open set $\widetilde{Y}$ containing Y.We let $\widetilde{\partial}(Y)$ denote the open boundary $\widetilde{Y} \setminus Y$ of Y.

For $x, y \in X$ we write $x \leq y$ when $\overline{\{x\}} \subseteq \overline{\{y\}}$, and $x < y$ when $x \leq y$ and $x \neq y$. For each $x \in X$, we denote by $\mathrm{Pr}(x)$ the set of all $y \in X$ such that $x < y$ and that no $z \in X$ satisfies $x < z < y$. We write $y \to x$ when $y \in \mathrm{Pr}(x)$. The following lemma is straightforward

Lemma 2.1. *For $x \in X$, the following hold:*

1. *The set $\mathrm{Pr}(x)$ coincides with the set of all closed points of $\widetilde{\partial}(\{x\})$.*
2. *We have $\widetilde{\partial}(\{x\}) = \bigcup_{y \in \mathrm{Pr}(x)} \widetilde{\{y\}}$, and consequently $\widetilde{\partial}(\{x\})$ is open.*
3. *An element $y \in X$ satisfies $x < y$ if and only if there exists a finite sequence $(z_k)_{k=1}^n$ in X such that $z_{k+1} \in \mathrm{Pr}(z_k)$ for $k = 1, \ldots, n-1$ where $z_1 = x$, $z_n = y$.*

We call a sequence $(z_k)_{k=1}^n$ as in Lemma 2.1(3) a *path* from y to x. We denote by $\mathrm{Path}(y, x)$ the set of paths from y to x. Thus Lemma 2.1(3) can be rephrased that $x, y \in X$ satisfies $x < y$ if and only if there exists a path from y to x. Such a path is unique if X is an accordion space, but in general not unique. Two $x, y \in X$ satisfies $y \in \mathrm{Pr}(x)$ if and only if (x, y) is a path from y to x, and in this case, there are no other paths.

The space X is called an *accordion space* if for each $x \in X$ there are at most two elements $z \in X$ satisfying $x \to z$ or $z \to x$, and if there are exactly two elements $x \in X$ for which there is exactly one element $z \in X$ satisfying $x \to z$ or $z \to x$. If X is linear, that is, if $X = \{x_1, \ldots, x_n\}$ with $x_n \to \cdots \to x_2 \to x_1$, then X is an accordion space.

3. Filtered K-theory

A *C^*-algebra A over X* is a C^*-algebra A equipped with a infima- and suprema-preserving map $\mathbb{O}(X) \to \mathbb{I}(A), U \to A(U)$ mapping open subsets in X to ideals in A. A *-homomorphism $\varphi\colon A \to B$ for C^*-algebras A and B over X is called *X-equivariant* if $\varphi(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$. Let $\mathbb{LC}(X)$ denote the set of locally closed subsets of X, i.e., subsets of the form $U \setminus V$ with U and V open subsets of X satisfying $V \subseteq U$. For $Y \in \mathbb{LC}(X)$, and $U, V \in \mathbb{O}(X)$ satisfying that $Y = U \setminus V$ and $U \supseteq V$, we define $A(Y)$ as $A(Y) = A(U)/A(V)$, which up to natural isomorphism is independent of the choice of U and V (see [11, Lemma 2.15]).

For a C^*-algebra A over X, $\mathrm{FK}_Y(A)$ is defined as $\mathrm{K}_*(A(Y))$ for all $Y \in \mathbb{LC}(X)$. We write $\mathrm{FK}_Y^i(A)$ for $\mathrm{K}_i(A(Y))$. Ralf Meyer and Ryszard Nest constructed in [10] C^*-algebras R_Y over X satisfying that the functors FK_Y and $\mathrm{KK}_*(X; R_Y, -)$ are equivalent.

In their definition of filtered K-theory FK, Meyer-Nest consider the $\mathbb{Z}/2$-graded category $\mathcal{NT}_*$ with objects $\mathbb{LC}(X)$ and morphisms

$$\mathrm{Nat}(\mathrm{FK}_Y, \mathrm{FK}_Z) \cong \mathrm{KK}_*(X; R_Z, R_Y)$$

between Y and Z, where $\mathrm{Nat}(\mathrm{FK}_Y, \mathrm{FK}_Z)$ denotes the set of natural transformations from the functor FK_Y to the functor FK_Z. The target category of FK is the category of graded modules over $\mathcal{NT}_*$, i.e., $\mathbb{Z}/2$-graded additive functors $\mathcal{NT}_* \to \mathfrak{Ab}^{\mathbb{Z}/2}$, hence $\mathrm{FK}(A)$ consists of the groups $\mathrm{FK}_Y(A)$ together with the natural transformations $\mathrm{FK}_Y(A) \to \mathrm{FK}_Z(A)$. To ease notation in the definitions to follow, we instead consider the category $\mathcal{NT}$ with objects $\mathbb{LC}(X) \times \{0,1\}$ and morphisms $\mathrm{Nat}(\mathrm{FK}_Y^j, \mathrm{FK}_Z^k) \cong \mathrm{KK}_0(X; \mathrm{S}^k R_Z, \mathrm{S}^j R_Y)$ between (Y,j) and (Z,k). The category of modules over $\mathcal{NT}$, i.e., additive functors $\mathcal{NT} \to \mathfrak{Ab}$, is equivalent to the target category of FK. Hence, in our notation,

$$\mathrm{FK}\colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{NT}).$$

Definition 3.1. Let $Y \in \mathbb{LC}(X)$, $U \subseteq Y$ open and set $C = Y \setminus U$. Such a pair (U,C) is called a *boundary pair.* The natural transformations occuring in the six-term exact sequence in K-theory for the distinguished ideal associated to $U \subseteq Y$ are denoted by i_U^Y, r_Y^C and δ_C^Y:

$$\begin{tikzcd}
\mathrm{FK}_U \arrow[rr, "i_U^Z"] & & \mathrm{FK}_Y \arrow[dl, "r_Y^C"] \\
& \mathrm{FK}_C \arrow[ul, "\circ" marking, "\delta_C^U"] &
\end{tikzcd}$$

They correspond to the $\mathrm{KK}(X)$-classes of $R_Y \twoheadrightarrow R_U$, $R_C \hookrightarrow R_Y$, and $R_C \hookrightarrow R_Y \twoheadrightarrow R_U$, respectively.

The following relations among the natural transformations acting on FK were established in [2].

Proposition 3.2. *In the category $\mathcal{NT}$, the following relations hold. By U, V, Y, C and D we denote generic elements of $\mathbb{LC}(X)$.*

(1) *For every $Y \in \mathbb{LC}(X)$,*

$$i_Y^Y = r_Y^Y = \mathrm{id}_Y .$$

(2) *If $Y \sqcup Z$ is a topologically disjoint union of subsets $Y, Z \in \mathbb{LC}(X)$, then*

$$r_{Y \cup Z}^Y i_Y^{Y \cup Z} + r_{Y \cup Z}^Z i_Z^{Y \cup Z} = \mathrm{id}_{Y \cup Z} .$$

In particular, the empty set $\emptyset$ is a zero object.

(3) *For open subsets $U \subseteq V \subseteq Y$,*

$$i_U^V i_V^Y = i_U^Y .$$

(4) *For closed subsets $C \subseteq D \subseteq Y$,*

$$r_Y^D r_D^C = r_Y^C .$$

(5) *Whenever $U \subseteq Y$ is open and $C \subseteq Y$ is closed,*

$$i_U^Y r_Y^C = r_U^{U \cap C} i_{U \cap C}^C .$$

(6) *Let (U,C) be a boundary pair in $\mathcal{NT}$ and define $Y = U \cup C$.*
    (i) *Let $C' \subseteq C$ be a relatively open subset. Then $U \cup C'$ is relatively open in $U \cup C$, the set C' is relatively closed in $U \cup C'$, and we have*

$$i_{C'}^C \delta_C^U = \delta_{C'}^U .$$

    (ii) *Let $U' \subseteq U$ be a relatively closed subset. Then $U' \cup C$ is relatively closed in $U \cup C$, the set U' is relatively open in $U' \cup C$, and*

$$\delta_C^U r_U^{U'} = \delta_C^{U'} .$$

(iii) *Let U' be a subset of U with the property that $U' \cup C$ is relatively open in $U \cup C$. Then U' is relatively open in U and in $U' \cup C$, and we have*

$$\delta_C^{U'} i_{U'}^U = \delta_C^U.$$

(iv) *Let C' be a subset of C with the property that $U \cup C'$ is relatively closed in $U \cup C$. Then C' is relatively closed in C and in $U \cup C'$, and*

$$r_C^{C'} \delta_{C'}^U = \delta_C^U.$$

(7) *Let (U, C) and (U', C') be boundary pairs in $\mathcal{NT}$ with $U \cup C = U' \cup C'$, and such that U is an open subset of U' and C' is a closed subset of C. Then*

$$\delta_C^U i_U^{U'} = r_C^{C'} \delta_{C'}^{U'}.$$

Remark 3.3. The vanishing of consecutive maps in six-term sequences associated to distinguished subquotient inclusions follows from the above relations.

Definition 3.4. Let $\mathcal{ST}$ be the universal preadditive category with generators as in Definition 3.1 and relations as in Proposition 3.2.

There is a canonical additive functor $\mathcal{ST} \to \mathcal{NT}$ which is an isomorphism in all examples which have been investigated so far, including accordion spaces (see [2,10]). We believe that this is also true for the more general UPP spaces defined in the following but do not give a proof here.

Let $\mathfrak{F}_{\mathcal{ST}} \colon \mathfrak{Mod}(\mathcal{NT}) \to \mathfrak{Mod}(\mathcal{ST})$ be the induced functor.

Definition 3.5. We define *concrete filtered* K*-theory* $\mathrm{FK}_{\mathcal{ST}} \colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{ST})$ as the composition $\mathfrak{F}_{\mathcal{ST}} \circ \mathrm{FK}$.

Definition 3.6. An $\mathcal{NT}$-module M is called *exact* if for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the sequence

$$\begin{tikzcd}
M(U,0) \arrow[r, "i"] & M(Y,0) \arrow[r, "r"] & M(Y \setminus U,0) \arrow[d, "\delta"] \\
M(Y \setminus U,1) \arrow[u, "\delta"] & M(Y,1) \arrow[l, "r"] & M(U,1) \arrow[l, "i"]
\end{tikzcd}$$

is exact. An $\mathcal{NT}$-module M is called *real-rank-zero-like* if for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the map $\delta \colon M(Y \setminus U, 0) \to M(U, 1)$ vanishes.

In the same way, we define exact $\mathcal{ST}$-modules and real-rank-zero-like $\mathcal{ST}$-modules.

Clearly, for a (real rank zero) C^*-algebra A over X, the modules $\mathrm{FK}(A)$ and $\mathrm{FK}_{\mathcal{ST}}(A)$ are exact (and real-rank-zero-like).

4. Sheafs

In this section we introduce sheaves and cosheaves and recall that it suffices to specify them on a basis for the topology.

Let X be an arbitrary topological space with topology $\mathbb{O}$. Let $\mathbb{B}$ be a basis for the topology on X. The sets $\mathbb{B}$ and $\mathbb{O}$ are partially ordered by inclusion.

Definition 4.1. A *presheaf on* $\mathbb{O}$ is a contravariant functor $M \colon \mathbb{O} \to \mathfrak{Ab}$. It is a *sheaf on* $\mathbb{O}$ if, for every open set U and every open covering $(U_j)_{j \in J}$ of U, the sequence

$$0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_j\cap U_k})-M(i_{U_k}^{U_j\cap U_k})\right)} \prod_{j,k\in J} M(U_j\cap U_k)$$

is exact.

More generally, a *presheaf on* $\mathbb{B}$ is a contravariant functor $M\colon \mathbb{B}\to\mathfrak{Ab}$. It is a *sheaf on* $\mathbb{B}$ if, for every open set $U\in\mathbb{B}$, every open covering $(U_j)_{j\in J}$ of U with $U_i\in\mathbb{B}$ and every open coverings $(U_{jkl})_{l\in L_{jk}}$ of $U_j\cap U_k$ with $U_{jkl}\in\mathbb{B}$, the sequence

$$0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_{jkl}})-M(i_{U_k}^{U_{jkl}})\right)} \prod_{j,k\in J}\prod_{l\in L_{jk}} M(U_{jkl}) \tag{4.2}$$

is exact. There is an obvious notion of morphism for sheafs. We denote by $\mathfrak{Sh}(\mathbb{B})$ the category of sheafs on $\mathbb{B}$.

Lemma 4.3. *The restriction functor* $\mathfrak{Sh}(\mathbb{O})\to\mathfrak{Sh}(\mathbb{B})$ *is an equivalence of categories.*

Proof. This is a well-known fact in algebraic geometry (see, for instance the encyclopedic treatment in [14, Lemma 009O]). We confine ourselves on mentioning that (4.2) provides a formula for computing $M(U)$ for an arbitrary open subset U. □

Definition 4.4. A *precosheaf on* $\mathbb{O}$ is a covariant functor $M\colon\mathbb{O}\to\mathfrak{Ab}$. It is a *cosheaf on* $\mathbb{O}$ if, for every open set U and every open covering $(U_j)_{j\in J}$ of U, the sequence

$$\bigoplus_{j,k\in J} M(U_j\cap U_k) \xrightarrow{\left(M(i_{U_j\cap U_k}^{U_j})-M(i_{U_j\cap U_k}^{U_k})\right)} \bigoplus_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U})\right)} M(U)\longrightarrow 0. \tag{4.5}$$

is exact.

More generally, a *precosheaf on* $\mathbb{B}$ is a covariant functor $M\colon\mathbb{B}\to\mathfrak{Ab}$. It is a *cosheaf on* $\mathbb{B}$ if, for every open set $U\in\mathbb{B}$, every open covering $(U_j)_{j\in J}$ of U with $U_i\in\mathbb{B}$ and every open coverings $(U_{jkl})_{l\in L_{jk}}$ of $U_j\cap U_k$ with $U_{jkl}\in\mathbb{B}$, the sequence

$$\bigoplus_{j,k\in J}\bigoplus_{l\in L_{jk}} M(U_{jkl}) \xrightarrow{\left(M(i_{U_{jkl}}^{U_j})-M(i_{U_{jkl}}^{U_k})\right)} \bigoplus_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U})\right)} M(U)\longrightarrow 0. \tag{4.6}$$

is exact. There is an obvious notion of morphism for cosheafs. We denote by $\mathfrak{CoSh}(\mathbb{B})$ the category of cosheafs on $\mathbb{B}$.

Lemma 4.7. *The restriction functor* $\mathfrak{CoSh}(\mathbb{O})\to\mathfrak{CoSh}(\mathbb{B})$ *is an equivalence of categories.*

Proof. This statement in the dual of Lemma 4.3 and follows in an analogous way. Again, (4.6) can be used to compute $M(U)$ for an arbitrary open subset U. □

With regard to the next section we remark that every finite T_0-space (more generally every Alexandrov space) comes with canonical bases for the open subsets, namely $\left\{\widetilde{\{x\}}\mid x\in X\right\}$, and for the closed subsets: $\left\{\overline{\{x\}}\mid x\in X\right\}$.

Lemma 4.8. *Let X be a finite T_0-space and let S be a pre(co)sheaf on the basis* $\mathbb{B}=\left\{\widetilde{\{x\}}\mid x\in X\right\}$. *Then S is a (co)sheaf.*

Proof. This follows from the observation that, in the basis $\mathbb{B}$ there are no non-trivial coverings, that is, if $\mathcal{U}$ is a covering of U then $U \in \mathcal{U}$. ☐

5. On the ordering of $\mathrm{K}_0(A)$

The notion of ordered filtered K-theory has been introduced by Eilers-Restorff-Ruiz in [9] to classify real rank zero graph algebras. In this section, we recall their definition and state some useful facts.

For a C^*-algebra A, an element $[p]_0$ in $\mathrm{K}_0(A)$ where p is a projection in $M_n(A)$ for some n is called *positive.* The *positive cone* $\mathrm{K}_0(A)^+$ consists of all positive elements in $\mathrm{K}_0(A)$.

For two C^*-algebras A and B, a group homomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called *positive* if $\varphi(\mathrm{K}_0(A)^+) \subseteq \mathrm{K}_0(B)^+$, and a group isomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called an *order-isomorphism* if $\varphi(\mathrm{K}_0(A)^+) = \mathrm{K}_0(B)^+$.

For C^*-algebras A and B over the space X, a $\mathcal{ST}$-momorphism $\varphi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is called *positive* if the induced maps $\mathrm{FK}^0_Y(A) \to \mathrm{FK}^0_Y(B)$ are positive for all $Y \in \mathbb{LC}(X)$, and an isomorphism is called an *order-isomorphism* if the induced isomomorphisms are order-isomorphisms. For reductions of filtered K-theory, the same definition applies.

In [6, 3.14], Lawrence G. Brown and Gert K. Pedersen showed that given an extension $I \hookrightarrow A \twoheadrightarrow A/I$ of C^*-algebras, the C^*-algebra A has real rank zero if and only if I and A/I have real rank zero and projections in A/I lift to projections in A/I. As real rank zero passes to matrices, we see that for a real rank zero C^*-algebra A and an ideal I in A, the induced map $\mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$ surjects the positive cone $\mathrm{K}_0(A)^+$ onto the positive cone $\mathrm{K}_0(A/I)^+$.

We are indebted to Mikael Rørdam for the elegant proof of the following lemma. As a consequence of the lemma, if a real rank zero C^*-algebra A can be written $A = I_1 + I_2 + \cdots + I_n$ with $I_1, \ldots, I_n$ ideals in A, then the induced map $\mathrm{K}_0(I_1) \oplus \cdots \oplus \mathrm{K}_0(I_n) \to \mathrm{K}_0(A)$ surjects the direct sum of the positive cones $\mathrm{K}_0(I_1)^+ \oplus \cdots \oplus \mathrm{K}_0(I_n)^+$ onto the positive cone $\mathrm{K}_0(A)^+$.

Lemma 5.1. *Let A be a real rank zero C^*-algebra and let I and J be (closed, two-sided) ideals in A satisfying $I + J = A$. Then any projection p in A can be written $p = q + q'$ with q a projection in I and q' a projection in J.*

Proof. Let p a projection in A be given and write $p = a + b$ with $a \in I$ and $b \in J$. We may assume that $a = pap$ and $b = pbp$. As A has real rank zero, the hereditary subalgebra pIp has an approximate unit of projections, so there exists a projection q in pIp satisfying $\|a - aq\| < \frac{1}{2}$. Since $q = pqp$, $q \leq p$ and we may define a projection q' as $q' = p - q$. Now, $q' = q'pq' = q'aq' + q'bq'$ with $q'bq' \in J$, so $\mathrm{dist}(q, J) \leq \|q'aq'\| < 1$, hence $q' + J$ is a projection in A/J of norm strictly less than 1, ergo $q' + J = J$. ☐

6. Filtered K-theory restricted to canonical base

In this section, the functor $\mathrm{FK}_{\mathcal{B}}$ and the notions of UPP spaces and BDP spaces are introduced.

The following lemma is straightforward to verify.

Lemma 6.1. *For a finite T_0-space X the following conditions are equivalent.*

- *There are no elements a, b, c, d in X with $a < b < d$, $a < c < d$ and neither $b \leq c$ nor $c \leq b$.*
- *In the Hasse diagram associated to the specialisation order on X, any two elements are connected by at most one path of directed edges.*
- *For all $x, y \in X$ with $x \to y$, $\widetilde{\{x\}} \cup \overline{\{y\}} \in \mathbb{LC}(X)$.*
- *For all $x \in X$, $\widetilde{\partial}(\{x\}) = \coprod_{y \to x} \widetilde{\{y\}}$.*
- *For all $x \in X$, $\overline{\partial}(\{x\}) = \coprod_{x \to y} \overline{\{y\}}$.*

Definition 6.2. A finite T_0-space X is called *UPP (unique path property)* if it satisfies the equivalent conditions specified in Lemma 6.1.

Let X be a UPP space.

Definition 6.3. Let $\mathcal{B}$ denote the universal preadditive category generated by objects $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $r_{\overline{x}_1}^{\overline{y}_1}$, $\delta_{\overline{y}_1}^{\widetilde{x}_0}$ and $i_{\widetilde{x}_0}^{\widetilde{y}_0}$ when $x \to y$, subject to the relations

$$\sum_{x \to y} r_{\overline{x}_1}^{\overline{y}_1} \delta_{\overline{y}_1}^{\widetilde{x}_0} = \sum_{z \to x} \delta_{\overline{x}_1}^{\widetilde{z}_0} i_{\widetilde{z}_0}^{\widetilde{x}_0} \tag{6.4}$$

for all $x \in X$.

Lemma 6.5. *In the category $\mathcal{ST}$, we have the relation*

$$\sum_{x \to y} r_{\overline{\{x\}}}^{\overline{\{y\}}} \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} = \sum_{z \to x} \delta_{\overline{\{x\}}}^{\widetilde{\{z\}}} i_{\widetilde{\{z\}}}^{\widetilde{\{x\}}}$$

for all $x \in X$.

Proof. Since X is a UPP space, the collections $\left(\overline{\{y\}}\right)_{x \to y}$ and $\left(\widetilde{\{z\}}\right)_{z \to x}$ are disjoint, respectively. Hence the desired relation simplifies to

$$r_{\overline{\{x\}}}^{\overline{\partial}\{x\}} \delta_{\overline{\partial}\{x\}}^{\widetilde{\{x\}}} = \delta_{\overline{\{x\}}}^{\widetilde{\partial}\{x\}} i_{\widetilde{\partial}\{x\}}^{\widetilde{\{x\}}},$$

which follows from Proposition 3.2(7). □

The previous lemma allows us to define an additive functor $\mathcal{B} \to \mathcal{ST}$ by $\overline{x}_1 \mapsto (\overline{\{x\}}, 1)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let

$$\mathfrak{F}_{\mathcal{B}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$$

denote the induced functor. Define *filtered* K*-theory restricted to the canonical base*, $\mathrm{FK}_{\mathcal{B}}$, as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{B}}$.

Definition 6.6. A $\mathcal{B}$-module M is called *exact* if the sequence

$$M(\overline{x}_1) \xrightarrow{\begin{pmatrix} r_{\overline{x}_1}^{\overline{y}_1} & -\delta_{\overline{x}_1}^{\widetilde{z}_0} \end{pmatrix}} \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta_{\overline{y}_1}^{\widetilde{x}_0} \\ i_{\widetilde{z}_0}^{\widetilde{x}_0} \end{pmatrix}} M(\widetilde{x}_0) \tag{6.7}$$

is exact for all $x \in X$.

Lemma 6.8. *If M is an exact $\mathcal{ST}$-module, then $\mathfrak{F}_{\mathcal{B}}(M)$ is an exact $\mathcal{B}$-module.*

In particular, if A is a C^*-algebra over X, then the $\mathcal{B}$-module $\mathrm{FK}_{\mathcal{B}}(A)$ *is exact.*

Proof. Using again that the collections $\left(\overline{\{y\}}\right)_{x\to y}$ and $\left(\widetilde{\{z\}}\right)_{z\to x}$ are respectively disjoint, it suffices to prove exactness of the sequence

$$M(\overline{\{x\}},1)\xrightarrow{\begin{pmatrix} r^{\overline{\partial}\{x\}}_{\overline{\{x\}}} & -\delta^{\widetilde{\partial}\{x\}}_{\overline{\{x\}}}\end{pmatrix}} M(\overline{\partial}\{x\},1)\oplus M(\widetilde{\partial}\{x\},0)\xrightarrow{\begin{pmatrix} \delta^{\widetilde{\{x\}}}_{\overline{\partial}\{x\}} \\ i^{\widetilde{\{x\}}}_{\widetilde{\partial}\{x\}}\end{pmatrix}} M(\widetilde{\{x\}},0),$$

which follows from a diagram chase through the commutative diagram

$$\begin{tikzcd}
M(\{x\},1) \arrow[r] \arrow[d, equal] & M(\overline{\{x\}},1) \arrow[r] \arrow[d, "\circ" marking] & M(\overline{\partial}\{x\},1) \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & M(\{x\},0) \arrow[d, equal] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(\widetilde{\partial}\{x\},0) \arrow[r] & M(\widetilde{\{x\}},0) \arrow[r] & M(\{x\},0)
\end{tikzcd}$$

whose rows are exact. □

Definition 6.9. A UPP space X is called *BDP* if it satisfies the following *boundary decomposition property*: for all boundary pairs (U,C) in X,

$$\delta^U_C=\sum_{x\to y,x\in U,y\in C} r^{\overline{\{y\}}\cap C}_{C}\; i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\; r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}}\; i^{U}_{\widetilde{\{x\}}\cap U}.$$

holds in the category $\mathcal{ST}$.

Theorem 6.10. *Let X be a BDP space. The functor*

$$\mathfrak{F}_{\mathcal{B}}\colon \mathfrak{Mod}(\mathcal{ST})\to\mathfrak{Mod}(\mathcal{B})$$

restricts to an equivalence between the category of exact real-rank-zero-like $\mathcal{ST}$-modules and the category of exact $\mathcal{B}$-modules.

For C^*-algebras A and B over X with real rank zero, an $\mathcal{ST}$-module homomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A)\to\mathrm{FK}_{\mathcal{ST}}(B)$ is an order-isomorphism if and only if $\mathfrak{F}_{\mathcal{B}}(\Phi)$ is.

A proof of this theorem is given after the following remark.

Remark 6.11. The invariant $\mathrm{FK}_{\mathcal{B}}$ is only defined for UPP spaces as the boundary map $\delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}$ only exists when $\overline{\{y\}}\cup\widetilde{\{x\}}$ belongs to $\mathbb{LC}(X)$. Also, the invariant $\mathrm{FK}_{\mathcal{B}}$ is most likely only sufficient for BDP spaces as for non-BDP spaces not all boundary maps can be recovered from $\mathrm{FK}_{\mathcal{B}}$.

Proof of Theorem 6.10. We will explicitly define a functor from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let an exact $\mathcal{B}$-module N be given. We will define an $\mathcal{ST}$-module M. We begin in the obvious way: For $x\in X$, let $M(\overline{\{x\}},1)=N(\overline{x}_1)$ and $M(\widetilde{\{x\}},0)=N(\widetilde{x}_0)$. Similarly, for $x\to y$, we define the even component of $i^{\widetilde{\{y\}}}_{\widetilde{\{x\}}}$ to be $i^{\widetilde{y}_0}_{\widetilde{x}_0}$, the odd component of $r^{\overline{\{y\}}}_{\overline{\{x\}}}$ to be $r^{\overline{y}_1}_{\overline{x}_1}$, and the odd-to-even component of $\delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}$ to be $\delta^{\widetilde{x}_0}_{\overline{y}_1}$. This makes sure that, finally, we will have $\mathfrak{F}_{\mathcal{B}}(M)=N$. Also, we of course define $\delta^U_C\colon M(C,0)\to M(U,1)$ to be zero for every boundary pair (U,C) so that M will be real-rank-zero-like.

For $x \geq y$, let $x \to x_1 \to x_2 \to \cdots \to x_n \to y$ be the unique path from x to y. Define the even component of $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ to be the composition $i_{\widetilde{x}_0}^{\widetilde{x_1}_0} i_{\widetilde{x_1}_0}^{\widetilde{x_2}_0} \cdots i_{\widetilde{x_n}_0}^{\widetilde{y}_0}$ and the odd component of $r_{\overline{\{x\}}}^{\overline{\{y\}}}$ as the composition $r_{\overline{x}_1}^{\overline{x_1}_1} r_{\overline{x_1}_1}^{\overline{x_2}_1} \cdots r_{\overline{x_n}_1}^{\overline{y}_1}$. In case of $x = y$, this specifies to $i_{\widetilde{x}_0}^{\widetilde{x}_0} = \mathrm{id}_{M(\widetilde{\{x\}},0)}$ and $r_{\overline{x}_1}^{\overline{x}_1} = \mathrm{id}_{M(\overline{\{x\}},1)}$. If we have $x \to y$, then these definitions coincide with the ones we gave before.

We observe that the groups $M(\widetilde{\{x\}}, 0)$ with the maps $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ define a precosheaf on $\mathbb{B} = \left\{\widetilde{\{x\}} \mid x \in X\right\}$. By Lemma 4.8 it is in fact a cosheaf. We can therefore apply Lemma 4.7 and obtain groups $M(U, 0)$ for all sets U and maps $i_U^V \colon M(U, 0) \to M(V, 0)$ for open sets $U \subseteq V$ which fulfill the relations (1) and (3) in Proposition 3.2.

For an arbitrary locally closed subset $Y \in \mathbb{LC}(X)$ we write $Y = V \setminus U$ with open sets $U \subseteq V$ and define $M(Y, 0)$ as the cokernel of the map $i_U^V \colon M(U, 0) \to M(V, 0)$. That this definition does not depend on the choice of U and V can be seen in a way similar to the proof of [11, Lemma 2.15] using that pushouts of abelian groups preserve cokernels. We obtain maps $r_V^Y \colon M(V, 0) \to M(Y, 0)$ for every open set V with relatively closed subset $Y \subseteq V$ such that the following holds: If $Y \in \mathbb{LC}(X)$ can be written as differences $V_i \setminus U_i$ of open sets for $i \in \{1, 2\}$ such that $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$, then the diagram

$$\begin{tikzcd}
M(U_1,0) \arrow[r, "i"] \arrow[d, "i"] & M(V_1,0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(Y,0) \arrow[d, equal] \\
M(U_2,0) \arrow[r, "i"] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y,0)
\end{tikzcd} \tag{6.12}$$

commutes.

For a relatively open subset $U \subseteq Y \in \mathbb{LC}(X)$ we obtain a map $i_U^Y \colon M(U, 0) \to M(Y, 0)$ using the diagram

$$\begin{tikzcd}
M(\widetilde{\partial} U,0) \arrow[r, "i"] \arrow[d, "i"] & M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[d, dotted, "i"] \\
M(\widetilde{\partial} Y,0) \arrow[r, "i"] & M(\widetilde{Y},0) \arrow[r, two heads, "r"] & M(Y,0).
\end{tikzcd} \tag{6.13}$$

It is easy to check that this map coincides with the previously defined one in case Y is open.

We find that, for $Y_i \in \mathbb{LC}(X)$ with $Y_1 \subseteq Y_2$ open, and $Y_i = V_i \setminus U_i$ for $i \in \{1, 2\}$ and open sets U_i, V_i such that $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$, the diagram

$$\begin{tikzcd}
M(U_1,0) \arrow[r, "i"] \arrow[d, "i"] & M(V_1,0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(Y_1,0) \arrow[d, "i"] \\
M(U_2,0) \arrow[r, "i"] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y_2,0)
\end{tikzcd} \tag{6.14}$$

commutes. We know this already for the left-hand square. For the right-hand square, it can be seen as follows: since V_1 is covered by U_1 and $\widetilde{Y_1}$, it suffices to check commutativity on the images $i_{U_1}^{V_1}(M(U_1))$ and $i_{\widetilde{Y_1}}^{V_1}(M(\widetilde{Y_1}))$. On $i_{U_1}^{V_1}(M(U_1))$

both compositions vanish. On the image of $M(\widetilde{Y_1})$, commutativity follows from (6.12) and (6.13) considering the diagram

$$\begin{tikzcd}
M(\widetilde{Y_1},0) \arrow[r, "i"'] \arrow[rr, bend left, "r"] \arrow[d, "i"] & M(V_1,0) \arrow[r, two heads, "r"'] \arrow[d, "i"] & M(Y_1,0) \arrow[d, "i"] \\
M(\widetilde{Y_2},0) \arrow[r, "i"] \arrow[rr, bend right, "r"'] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y_2,0).
\end{tikzcd}$$

Now let $Y \in \mathbb{LC}(X)$, let U be a relatively open subset of Y and let $C = Y \setminus U$. Consider the diagram

$$\begin{tikzcd}
M(\widetilde{\partial}U,0) \arrow[r, "i"] \arrow[d, "i"] & M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[d, "i"] \\
M(\widetilde{\partial}Y,0) \arrow[r, "i"] \arrow[d, two heads, "r"] & M(\widetilde{Y},0) \arrow[r, two heads, "r"] \arrow[d, two heads, "r"] & M(Y,0) \arrow[d, two heads, dotted] \\
M(\widetilde{\partial}Y \setminus \widetilde{\partial}U,0) \arrow[r, "i"] & M(\widetilde{Y} \setminus \widetilde{U},0) \arrow[r, two heads, "r"] & M(C,0),
\end{tikzcd} \tag{6.15}$$

whose solid squares commute and whose rows and solid columns are exact. A diagram chase shows that there is a unique surjective map $r_Y^C\colon M(Y,0) \to M(C,0)$, as indicated by the dotted arrow, making the bottom-right square commute and making the right-hand column exact at $M(Y,0)$. Again, we can easily check that this map coincides with the previously defined one in case Y is open.

We have now defined the even part of the module M completely. It is straightforward to check the relations (3) and (4) in Proposition 3.2. We will now prove that the relation (5) holds as well.

For this purpose, fix $Y \in \mathbb{LC}(X)$, let $U \subseteq Y$ be open and let $C \subseteq Y$ be closed. Consider the diagram

$$\begin{tikzcd}
M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[r, "r"] \arrow[d, "i"] & M(U \cap C,0) \arrow[d, "i"] \\
M(\widetilde{Y},0) \arrow[r, "r"] & M(Y,0) \arrow[r, "r"] & M(C,0)
\end{tikzcd}$$

We would like to proof that the right hand square commutes. The left hand square commutes by definition of the map i_U^Y. Since $\widetilde{U} \cap C = U \cap C$, we can therefore assume without loss of generality that U and Y are open. Commutativity then follows from (6.14).

Next, we will convince ourselves that the relation (2) in Proposition 3.2 holds on the even part of M. Let $W = Y \sqcup Z$ be a topologically disjoint union of subsets $Y, Z \in \mathbb{LC}(X)$. Fix $w \in W$. Then $(w - wr_W^Z i_Z^W)r_W^Z = 0$ as $i_Z^W r_W^Z = \mathrm{id}_Z$. Hence there is $y \in Y$ with $yi_Y^W = w - wr_W^Z i_Z^W$. Applying r_W^Y shows $y = wr_W^Y$ as $i_Z^W r_W^Y = 0$. We get

$$w(r_W^Y i_Y^W + r_W^Z i_Z^W) = yi_Y^W + wr_W^Z i_Z^W = w.$$

We have shown that $r_W^Y i_Y^W + r_W^Z i_Z^W = \mathrm{id}_W$ as desired.

We have defined all even groups for the desired module M and the action of all transformations between them. We have checked all relations only involving transformations between even groups and verified exactness of $M(C,0) \to M(Y,0) \to M(U,0)$ for every boundary pair $Y = U \cup C$.

We intend to do the same for the odd part of the module M in an analogous way. We start out with the given data consisting of the groups $M(\overline{\{x\}},1)$ and the maps $r_{\overline{x}_1}^{\overline{y}_1}$, $x \to y$, extend this to a sheaf on the basis $\big\{\overline{\{x\}} \mid x \in X\big\}$ of closed sets and apply Lemma 4.3. Observing that every locally closed subset of X can be written as a difference of two nested closed sets and using the functoriality of the kernel of a group homomorphism, we define groups $M(\overline{Y},1)$ for all $Y \in \mathbb{LC}(X)$ and actions for all transformations between these odd groups. Using arguments analogous to the ones above, we can verify the relations (1) to (5) in Proposition 3.2 on the odd part of M. It remains to define the odd-to-even components of the boundary maps δ_C^U for all boundary pairs (U,C), which has only be done in the special case $U = \widetilde{x}$, $C = \overline{y}$ with $x \to y$. Our general definition for $\delta_C^U\colon M(C,1) \to M(U,0)$ is

$$\delta_C^U = \sum_{x\to y, x\in U, y\in C} r_C^{\overline{\{y\}}\cap C}\ i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\ i_{\widetilde{\{x\}}\cap U}^{U}. \tag{6.16}$$

Our next aim is to verify the relations in (6) and (7) Proposition 3.2. We begin with relation (6)(i). Let (U,C) be a boundary pair and let $C' \subseteq C$ be relatively open. We have by the relations (3) and (5) that

$$\begin{aligned} i_{C'}^{C}\ \delta_C^U &= \sum_{x\to y, x\in U, y\in C} i_{C'}^{C}\ r_C^{\overline{\{y\}}\cap C}\ i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\ i_{\widetilde{\{x\}}\cap U}^{U} \\ &= \sum_{x\to y, x\in U, y\in C} r_{C'}^{\overline{\{y\}}\cap C'}\ i_{\overline{\{y\}}\cap C'}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\ i_{\widetilde{\{x\}}\cap U}^{U}. \end{aligned}$$

Since C' is relatively open in C, $\overline{\{y\}} \cap C'$ is empty unless $y \in C'$. Therefore, the above sum equals

$$\delta_{C'}^U = \sum_{x\to y, x\in U, y\in C'} r_{C'}^{\overline{\{y\}}\cap C'}\ i_{\overline{\{y\}}\cap C'}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\ i_{\widetilde{\{x\}}\cap U}^{U}.$$

This shows relation (6)(i). The relation (6)(ii) follows similarly.

Next we will check relation (6)(iii). Let (U,C) be a boundary pair and let U' be a subset of U such that $U' \cup C$ is relatively open in $U \cup C$. This relative openness condition ensures that $x \to y$, $x \in U$, $y \in C$ implies $x \in U'$. Moreover, for $x \in U'$, we have $\widetilde{x} \cap U' = \widetilde{x} \cap U$. Hence we get

$$\begin{aligned} \delta_C^U &= \sum_{x\to y, x\in U', y\in C} r_C^{\overline{\{y\}}\cap C}\ i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\ i_{\widetilde{\{x\}}\cap U}^{U} \\ &= \sum_{x\to y, x\in U', y\in C} r_C^{\overline{\{y\}}\cap C}\ i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U'}\ i_{\widetilde{\{x\}}\cap U'}^{U'}\ i_{U'}^{U} = \delta_C^{U'}\ i_{U'}^{U}. \end{aligned}$$

Again, relation (6)(iv) follows in a similar way.

Now we turn to relation (7). Let (U,C) be a boundary pair and let $p \in C$ be a maximal point. Then $U' = U \cup \{p\}$ and $C' = C \setminus \{p\}$ form a boundary pair with $U \cup C = U' \cup C'$, $U \subseteq U'$ relatively open and $C' \subseteq C$ relatively closed. It suffices to verify relation (7) in the particular situation above, because *every* boundary pair

(U',C') with $U \cup C = U' \cup C'$, $U \subseteq U'$ relatively open and $C' \subseteq C$ relatively closed can be obtained from (U,C) by performing the above procedure finitely many times.

Since N is a $\mathcal{B}$-module, we have

$$\sum_{p \to y} r^{\overline{\{y\}}}_{\overline{\{p\}}}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}} = \sum_{x \to p} \delta^{\widetilde{\{x\}}}_{\overline{\{p\}}}\ i^{\widetilde{\{p\}}}_{\widetilde{\{x\}}}.$$

Multiplying from the right with $r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'}$ we get by the relations (5) and (1) that

$$\begin{aligned}
\sum_{p \to y} r^{\overline{\{y\}}}_{\overline{\{p\}}}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} &= \sum_{x \to p} \delta^{\widetilde{\{x\}}}_{\overline{\{p\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{x \to p, x \in U} \delta^{\widetilde{\{x\}}}_{\overline{\{p\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'}.
\end{aligned}$$

In the last step we have used that $\widetilde{x} \cap U'$ is empty for $x \to p$ with $x \notin U$ because U' is locally closed. Multiplying from the left with $r^{\overline{\{p\}}\cap C}_{C}\ \ i^{\overline{\{p\}}}_{\overline{\{p\}}\cap C}$, we then obtain

$$\begin{aligned}
&\sum_{x \to p, x \in U} r^{\overline{\{p\}}\cap C}_{C}\ \ i^{\overline{\{p\}}}_{\overline{\{p\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{p\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{p \to y} r^{\overline{\{p\}}\cap C}_{C}\ \ i^{\overline{\{p\}}}_{\overline{\{p\}}\cap C}\ r^{\overline{\{y\}}}_{\overline{\{p\}}}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{p \to y} r^{\overline{\{y\}}\cap\overline{\{p\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap\overline{\{p\}}\cap C}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{p \to y, y \in C'} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\delta^{U}_{C}\ i^{U'}_{U} &= \sum_{x \to y, x \in U, y \in C} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{x \to p, x \in U} r^{\overline{\{p\}}\cap C}_{C}\ \ i^{\overline{\{p\}}}_{\overline{\{p\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{p\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&\quad + \sum_{x \to y, x \in U, y \in C'} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{p \to y, y \in C'} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{p\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{p\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&\quad + \sum_{x \to y, x \in U, y \in C'} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} \\
&= \sum_{x \to y, x \in U', y \in C'} r^{\overline{\{y\}}\cap C}_{C}\ \ i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\ \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\ r^{\widetilde{\{x\}}\cap U'}_{\widetilde{\{x\}}}\ \ i^{U'}_{\widetilde{\{x\}}\cap U'} = r^{C'}_{C}\ \delta^{U'}_{C'}.
\end{aligned}$$

This finishes the verification of the relations in Proposition 3.2. Hence, M is indeed an $\mathcal{ST}$-module. To see that M is exact, it remains to show that the sequences $M(C,1) \xrightarrow{\delta^{U}_{C}} M(U,0) \xrightarrow{i^{Y}_{U}} M(Y,0)$ and $M(Y,1) \xrightarrow{r^{C}_{Y}} M(C,1) \xrightarrow{\delta^{U}_{C}} M(U,0)$ are exact for all boundary pairs (U,C) with $Y = U \cup C$.

Fix an element $x \in X$ and consider the commutative diagram

$$\begin{tikzcd}
M(\{x\},1) \arrow[r, "i"] \arrow[d, equal] & M(\overline{\{x\}},1) \arrow[r, "r"] \arrow[d, "\circ" marking] & M(\overline{\partial}\{x\},1) \arrow[d, "\circ" marking] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(\widetilde{\partial}\{x\},0) \arrow[r, "i"] & M(\widetilde{\{x\}},0)
\end{tikzcd}$$

Using exactness of the upper row and the fact that N was an exact $\mathcal{B}$-module, a diagram chase shows that the bottom row is exact. In a similar way, we see that the sequence

$$M(\overline{\{x\}},1) \to M(\overline{\partial}\{x\},0) \to M(\{x\},0).$$

is exact for every $x \in X$.

Next, let $Y \in \mathbb{LC}(X)$ and let $x \in Y$ be a closed point. Then $Y \cap \widetilde{\{x\}}$ is relatively closed in $\widetilde{\{x\}}$ because Y is locally closed. A diagram chase in the commutative diagram

$$\begin{tikzcd}
 & M(\widetilde{\partial}(x) \setminus (Y \cap \widetilde{\partial}(x)),0) \arrow[r, equal] \arrow[d, "i"] & M(\widetilde{\{x\}} \setminus (Y \cap \widetilde{\{x\}}),0) \arrow[d, "i"] \\
M(\{x\},1) \arrow[r, "\circ" marking] \arrow[d, equal] & M(\widetilde{\partial}\{x\},0) \arrow[r, "i"] \arrow[d, two heads, "r"] & M(\widetilde{\{x\}},0) \arrow[d, two heads, "r"] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(Y \cap \widetilde{\partial}\{x\},0) \arrow[r, "i"] & M(Y \cap \widetilde{\{x\}},0),
\end{tikzcd}$$

whose columns and first row are exact, yields exactness of the bottom row. By a diagram chase in the commutative diagram

$$\begin{tikzcd}
M(\{x\},1) \arrow[r, "\circ" marking] \arrow[d, equal] & M(Y \cap \widetilde{\partial}\{x\},0) \arrow[r, "i"] \arrow[d, "i"] & M(Y \cap \widetilde{\{x\}},0) \arrow[d, "i"] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(Y \setminus \{x\},0) \arrow[r, "i"] & M(Y,0)
\end{tikzcd}$$

using the exact cosheaf sequence (4.5) for the covering $(Y \setminus \{x\}, Y \cap \widetilde{\{x\}})$ of Y we obtain exactness of the bottom row. Notice that, using a further diagram chase, it is not hard to deduce the exactness of the cosheaf sequence for a relatively open covering of a locally closed set from the open case.

We have established the exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ for all boundary pairs (U,C) with C a singleton. Analogously, we find that $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ is exact whenever U is a singleton.

We will proceed by an inductive argument. Let $n \geq 1$ be a natural number and assume that exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ is proven for all boundary pair (U,C) for which C has at most n elements. Let (U,C) be a boundary pair such that C has $n+1$ elements. Write $Y = U \cup C$. Let $p \in C$ be a maximal point and set $U' = U \cup \{p\}$, $C' = C \setminus \{p\}$. Then (U', C') is a boundary pair. A diagram chase in the commutative diagram

$$\begin{tikzcd}
M(\{p\},1) \arrow[r, "i"] \arrow[d, equal] & M(C,1) \arrow[r, "r"] \arrow[d, "\circ" marking] & M(C',1) \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & M(\{p\},0) \arrow[d, equal] \\
M(\{p\},1) \arrow[r, "\circ" marking] & M(U,0) \arrow[r, "i"] \arrow[d, "i"] & M(U',0) \arrow[r, "r"] \arrow[d, "i"] & M(\{p\},0) \\
& M(Y,0) \arrow[r, equal] & M(Y,0),
\end{tikzcd}$$

whose rows and third column are exact, shows exactness of the second column. Again, exactness of $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ for all boundary pairs follows in a analogous manner. We conclude that M is an exact $\mathcal{ST}$-module.

Summing up, we have associated an exact real-rank-zero-like $\mathcal{ST}$-module with every exact $\mathcal{B}$-module. By the naturality of our constructions using kernels and cokernels we in fact obtain a functor G from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Let F be the restriction of the functor $\mathfrak{F}_{\mathcal{B}}$ to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Then the composition GF is equal to the identity functor on the category of exact $\mathcal{B}$-modules. It remains to show that FG is naturally isomorphic to the identity functor on the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. We will construct a natural $\mathcal{ST}$-module isomorphism $\eta_M \colon M \to (FG)(M)$. For $x \in X$ we have $M(\widetilde{\{x\}},0) = (FG)(M)(\widetilde{\{x\}},0)$ and $M(\overline{\{x\}},1) = (FG)(M)(\overline{\{x\}},1)$. Hence we set $\eta_M(\widetilde{\{x\}},0) = \mathrm{id}_{M(\widetilde{\{x\}},0)}$ and $\eta_M(\overline{\{x\}},1) = \mathrm{id}_{M(\overline{\{x\}},1)}$. Using the universal property of kernels and cokernels we obtain natural group homomorphisms $f_Y \colon M(Y,1) \to (FG)(M)(Y,1)$ and $g_Y \colon (FG)(M)(Y,0) \to M(Y,0)$ for every $Y \in \mathbb{LC}(X)$. An application of the five lemma shows that these are in fact isomorphisms. We can therefore define $\eta_M(Y,1) = f_Y$ and $\eta_M(Y,0) = (g_Y)^{-1}$.

Finally, we check that this collection of maps consitutes an $\mathcal{ST}$-module homomorphism, that is, the group homomorphism $\eta_M \colon M \to (FG)(M)$ intertwines the actions of the category $\mathcal{ST}$ on M and on $(FG)(M)$. By construction this is true for the transformations $(i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}},0)$, $(r_{\overline{\{x\}}}^{\overline{\{y\}}},1)$ and $\delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}$ for all $x,y \in X$ with $x \to y$. By Lemma 4.3 and Lemma 4.7 it is also true for the transformation $(i_U^V,0)$ for all open subset U,V of X with $U \subseteq V$ and for $(r_C^D,1)$ for all closed subsets C,D of X with $D \subseteq C$.

Let $V \subseteq X$ be open and let $Y \subseteq V$ be relativelty closed. Since $(r_V^Y,0)$ was defined as a natural projection onto a cokernel, our assertion holds for this transformation as well. Consequently, by (6.13) the assertion also follows for the transformation $(i_U^Y,0)$ for $Y \in \mathbb{LC}(X)$ and $U \subseteq Y$ relatively open. Finally (6.15) implies the assertion for the transformation r_Y^C with $Y \in \mathbb{LC}(X)$ and $C \subseteq Y$ relatively closed. We have shown that η intertwines the actions of all even transformations on the 0-parts of M and $(FG)(M)$. By analogous arguments the same follows for the actions of all even transformations on the 1-parts of M and $(FG)(M)$.

Our last step is to consider the action of a boundary transformation δ_C^U for a boundary pair (U,C). Since M and $(FG)(M)$ are real-rank-zero-like the 0-to-1 component of δ_C^U acts trivially on both modules. We have already seen that the assertion is true for the 1-to-0 component of δ_C^U in the specific case that $(U, C) = (\widetilde{\{x\}}, \overline{\{y\}})$ with $x \to y$. The general case then follows from (6.15) as X is BDP.

Finally, the assertion on positivity — i.e., that for real rank zero C^*-algebras A and B over X a $\mathcal{ST}$-morphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is an order-isomorphism if and only if $\mathfrak{F}_{\mathcal{B}}(\Phi)$ is — follows from Section 5 as our construction uses cokernels. □

Definition 6.17. Let X be a finite T_0-space. A boundary pair (U, C) in X is called *elementary* if U and C are connected and non-empty, U is open, C is closed and if, moreover, $U \subseteq \widetilde{C}$ and $C \subseteq \overline{U}$.

Lemma 6.18. *Let X be a UPP space with the property that every elementary boundary pair (U, C) in X is of the form $(\widetilde{\{x\}}, \overline{\{y\}})$ for two points x and y in X with $x \to y$. Then X is a BDP space.*

Proof. Let (U, C) be a boundary pair in X. We would like to show that the relation

$$\delta_C^U = \sum_{x\to y, x\in U, y\in C} r_C^{\overline{\{y\}}\cap C}\; i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\; i_{\widetilde{\{x\}}\cap U}^{U}$$

holds in the category $\mathcal{ST}$. We will reduce this statement to a special case using the relations listed in Proposition 3.2. Notice that, if we define

$$d_C^U = \sum_{x\to y, x\in U, y\in C} r_C^{\overline{\{y\}}\cap C}\; i_{\overline{\{y\}}\cap C}^{\overline{\{y\}}}\; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap U}\; i_{\widetilde{\{x\}}\cap U}^{U}$$

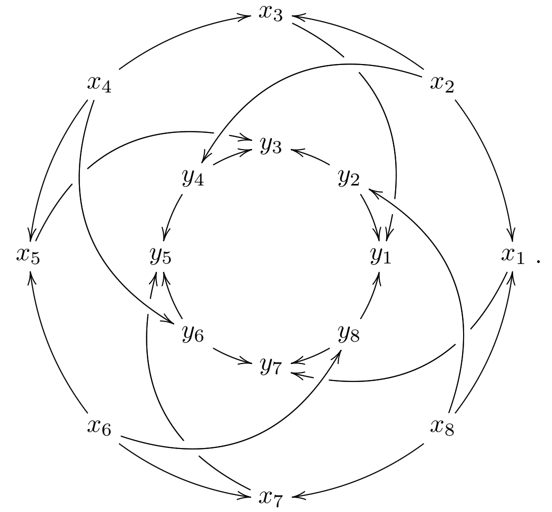
then, by the proof of Theorem 6.10, the relations (6) hold with d in place of δ.

Using the relations (2) and (6) we can thus assume without loss of generality that U and C are connected. Furthermore, it follows from the investigations in [2, §3.2] that using the relations (6) in Proposition 3.2 we can moreover assume that the boundary pair (U, C) is elementary. In this case, the assertion follows directly from our assumption. □

Corollary 6.19. *Let X be a finite T_0-space. Assume that the directed graph associated to X is a* forest, *i.e., it contains no undirected cycles. Then X is a BDP space.*

Proof. It is clear that, if the directed graph associated to X is a forest, then X is a UPP space. The assertion will be proved by contradiction using the previous lemma. Let (U, C) be an elementary boundary not of the form $(\widetilde{\{x\}}, \overline{\{y\}})$ for any $x, y \in X$. Choose a maximal element $c \in C$. Since $C \subseteq \overline{U}$, there is $u \in U$ with $u > c$. We can, moreover, assume that $u \to c$ because $U \cup C$ is locally closed and c is maximal in C. Since U is open and C is closed, we have $\widetilde{\{u\}} \subseteq U$ and $\overline{\{c\}} \subseteq C$. By our assumption, one of these inclusion must be strict. Without loss of generality we assume $\overline{\{c\}} \subsetneq C$. Choose $d \in C \setminus \overline{\{c\}}$. If $d \in \widetilde{\{u\}}$, then, since C is connected and $d \notin \overline{\{c\}}$, there is a cycle in X. If, on the other hand, $d \notin \widetilde{\{u\}}$, then, by $C \subseteq \overline{U}$, there is $v \in U$ with $v > d$. Using that U and C are connected, we again obtain a cycle in X. □

Remark 6.20. The above corollary applies, in particular, to accordion spaces. The condition of Lemma 6.18 can also be verified for various UPP spaces which are not forests—the smallest example being the so-called pseudocircle with four points. Consider, however, the sixteen-point space Q defined by the directed graph



Then Q is a UPP space that does not satisfy the condition in Lemma 6.18 as the subsets $U = \{x_1, x_2, \dots, x_8\}$ and $C = \{y_1, y_2, \dots, y_8\}$ give an elementary boundary pair (U, C) that does not satisfy $U = \widetilde{\{x\}}$ nor $C = \overline{\{y\}}$ for any $x, y \in X$. A simple computation shows that the boundary decomposition of δ_C^U holds in the category $\mathcal{NT}$. It appears, however, that it does not hold in $\mathcal{ST}$.

7. Reduced filtered K-theory

Let X be an arbitrary finite T_0-space. In this section we introduce a functor $\mathrm{FK}_{\mathcal{R}}$ which is equivalent to the reduced filtered K-theory defined by Gunner Restorff in [13].

Definition 7.1. Let $\mathcal{R}$ denote the universal preadditive category generated by objects $x_1, \widetilde{\partial x}_0, \widetilde{x}_0$ for all $x \in X$ and morphisms $\delta_{x_1}^{\widetilde{\partial x}_0}$ and $i_{\widetilde{\partial x}_0}^{\widetilde{x}_0}$ for all $x \in X$, and $i_{\widetilde{y}_0}^{\widetilde{\partial x}_0}$ when $y \to x$, subject to the relations

$$\delta_{x_1}^{\widetilde{\partial x}_0} i_{\widetilde{\partial x}_0}^{\widetilde{x}_0} = 0 \tag{7.2}$$

$$i_p i_{\widetilde{y(p)}_0}^{\widetilde{\partial x}_0} = i_q i_{\widetilde{y(q)}_0}^{\widetilde{\partial x}_0} \tag{7.3}$$

for all $x \in X$, all $y \in X$ satisfying $y > x$, and all paths $p, q \in \mathrm{Path}(y, x)$, where for a path $p = (z_k)_{k=1}^n$ in $\mathrm{Path}(y, x)$, we define $y(p) = z_2$, and

$$i_p = i_{\widetilde{z_n}_0}^{\widetilde{\partial z_{n-1}}_0} i_{\widetilde{\partial z_{n-1}}_0}^{\widetilde{z_{n-2}}_0} \cdots i_{\widetilde{z_3}_0}^{\widetilde{\partial z_2}_0} i_{\widetilde{\partial z_2}_0}^{\widetilde{z_2}_0}.$$

It is easy to see that the relations in $\mathcal{ST}$ corresponding to (7.2) and (7.3) hold. We can thus define an additive functor $\mathcal{R} \to \mathcal{ST}$ by $x_1 \mapsto (\{x\}, 1)$, $\widetilde{\partial x}_0 \mapsto (\widetilde{\partial}(x), 0)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let $\mathfrak{F}_{\mathcal{R}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{R})$ denote the induced functor. Define *reduced filtered* K*-theory*, $\mathrm{FK}_{\mathcal{R}}$ as the composition of FK with $\mathfrak{F}_{\mathcal{R}}$.

Definition 7.4. An $\mathcal{R}$-module M is called *exact* if the sequences

$$M(x_1) \xrightarrow{\delta} M(\widetilde{\partial} x_0) \xrightarrow{i} M(\widetilde{x}_0) \tag{7.5}$$

$$\bigoplus_{(p,q)\in \mathrm{DP}(x)} M(\widetilde{z(p,q)}_0) \xrightarrow{(i_p - i_q)_{(p,q)}} \bigoplus_{y\to x} M(\widetilde{y}_0) \xrightarrow{(i_{\widetilde{y}_0}^{\widetilde{\partial} x_0})} M(\widetilde{\partial} x_0) \longrightarrow 0 \tag{7.6}$$

are exact for all $x \in X$, where $\mathrm{DP}(x)$ denotes the set of pairs distinct paths (p,q) to x and from some commen element which is denoted $z(p,q)$.

Lemma 7.7. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module.Let Y be an open subset of X and let $(U_i)_{i\in I}$ be an open covering of Y. Then the following sequence is exact:*

$$\bigoplus_{i,j\in I} M(U_i \cap U_j, 0) \xrightarrow{(i_{U_i\cap U_j}^{U_i} - i_{U_i\cap U_j}^{U_j})} \bigoplus_{i\in I} M(U_i, 0) \xrightarrow{(i_{U_i}^{Y})} M(Y, 0) \longrightarrow 0.$$

Proof. Using an inductive argument as in [5, Proposition 1.3], we can reduce to the case that I has only two elements. In this case, exactness follows from a straightforward diagram chase using the exact six-term sequences of the involved ideal inclusions analogous to the one in the proof of Lemma 6.8. ☐

Corollary 7.8. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module and set $N = \mathfrak{F}_{\mathcal{R}}(M)$. Then N is an exact $\mathcal{R}$-module.*

Proof. We verify the exactness of the desired sequences in M. The sequence (7.5) is exact since it is part of the exact six-term sequence associated to the open inclusion $\widetilde{\partial}\{x\} \subseteq \widetilde{\{x\}}$.

To prove exactness of the sequence (7.6), we apply the previous lemma to $Y = \widetilde{\partial}\{x\}$ and get the exact sequence

$$\bigoplus_{y\to x, y'\to x} M(\widetilde{\{y\}} \cap \widetilde{\{y'\}}, 0) \xrightarrow{\left(i_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}}^{\widetilde{\{y\}}} - i_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}}^{\widetilde{\{y'\}}}\right)} \bigoplus_{y\to x} M(\widetilde{\{y\}}, 0)$$
$$\xrightarrow{\left(i_{\widetilde{\{y\}}}^{\widetilde{\partial}\{x\}}\right)} M(\widetilde{\partial}\{x\}, 0) \longrightarrow 0.$$

Another application of the previous lemma shows that $\bigoplus_{(p,q)\in\mathrm{DP}(x)} M(\widetilde{z(p,q)}, 0)$ surjects onto $\bigoplus_{y\to x, y'\to x} M(\widetilde{\{y\}} \cap \widetilde{\{y'\}}, 0)$ in a way making the obvious triangle commute. This establishes the exact sequence (7.6). ☐

Remark 7.9. If X is a UPP space, then the set $\mathrm{DP}(x)$ is empty for every $x \in X$. Hence, for an exact $\mathcal{R}$-module M, the map $(i_{\widetilde{y}_0}^{\widetilde{\partial} x_0})\colon \bigoplus_{y\to x} M(\widetilde{y}_0) \to M(\widetilde{\partial} x_0)$ is an isomorphism. In this sense, the groups $M(\widetilde{\partial} x_0)$ are redundant for an exact $\mathcal{R}$-module in case X is UPP.

8. An intermediate invariant

In this section, we define one more invariant, which, in a sense, can be thought of as a union or join of reduced filtered K-theory and filtered K-theory restricted to canonical base. It functions as an intermediate invariant towards filtered K-theory.

Let X be a UPP space.

Definition 8.1. Let $\mathcal{BR}$ denote the universal preadditive category generated by objects x_1, $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $i_{x_1}^{\overline{x}_1}$ for all $x \in X$ and $r_{\overline{x}_1}^{\overline{y}_1}$, $\delta_{\overline{y}_1}^{\widetilde{x}_0}$ and $i_{\widetilde{x}_0}^{\widetilde{y}_0}$ when $x \to y$, subject to the relations

$$\sum_{x \to y} r_{\overline{x}_1}^{\overline{y}_1} \delta_{\overline{y}_1}^{\widetilde{x}_0} = \sum_{z \to x} \delta_{\overline{x}_1}^{\widetilde{z}_0} i_{\widetilde{z}_0}^{\widetilde{x}_0} \tag{8.2}$$

for all $x \in X$ and

$$i_{x_1}^{\overline{x}_1} r_{\overline{x}_1}^{\overline{y}_1} = 0 \tag{8.3}$$

when $x \to y$.

As before, there is a canonical additive functor $\mathcal{BR} \to \mathcal{ST}$, inducing a functor $\mathfrak{F}_{\mathcal{BR}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{BR})$. Define $\mathrm{FK}_{\mathcal{BR}}$ as the composition of FK with $\mathfrak{F}_{\mathcal{BR}}$.

The category $\mathcal{B}$ embeds into $\mathcal{BR}$, and a forgetful functor $\mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{B})$ is induced.

Define an additive functor $\mathfrak{F}_{\mathcal{BR},\mathcal{R}} \colon \mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{R})$ by

$$M(\widetilde{\partial} x_0) = \bigoplus_{y \to x} M(\widetilde{y}_0)$$

and $i_{x_1}^{\widetilde{\partial} x_0} = (i_{x_1}^{\overline{x}_1} \delta_{\overline{x}_1}^{\widetilde{y}_0})$. One can check that this functor is well-defined.

Definition 8.4. An $\mathcal{BR}$-module M is called *exact* if the sequences

$$M(\overline{x}_1) \xrightarrow{\begin{pmatrix} r_{\overline{x}_1}^{\overline{y}_1} & -\delta_{\overline{x}_1}^{\widetilde{z}_0} \end{pmatrix}} \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta_{\overline{y}_1}^{\widetilde{x}_0} \\ i_{\widetilde{z}_0}^{\widetilde{x}_0} \end{pmatrix}} M(\widetilde{x}_0) \tag{8.5}$$

$$0 \longrightarrow M(x_1) \xrightarrow{i_{x_1}^{\overline{x}_1}} M(\overline{x}_1) \xrightarrow{(r_{\overline{x}_1}^{\overline{y}_1})} \bigoplus_{x \to y} M(\overline{y}_1) \tag{8.6}$$

are exact for all $x \in X$ and all $y \in X$ satisfying $x \to y$.

Lemma 8.7. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Then $\mathfrak{F}_{\mathcal{BR}}(M)$ is an exact $\mathcal{BR}$-module.*

Proof. The proof is similar to the proof of Lemma 6.8. □

Theorem 8.8. *Assume that X is a UPP space. Let M and N be exact $\mathcal{BR}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi \colon \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists an $\mathcal{BR}$-module homomorphism $\Phi \colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$, and if φ is an isomorphism then so is Φ.*

If $M = \mathrm{FK}_{\mathcal{BR}}(A)$ and $N = \mathrm{FK}_{\mathcal{BR}}(B)$ for C^*-algebras A and B over X with real rank zero, then Φ is an order-isomorphism if and only if φ is.

Proof. For $x \in X$, we define $\Phi_{x_1} = \varphi_{x_1}$ and $\Phi_{\widetilde{x}_0} = \varphi_{\widetilde{x}_0}$. In the following, we will define $\Phi_{\overline{x}_1}$ by induction on the partial order of X in a way such that the relations

$$r_{\overline{x}_1}^{\overline{y}_1} \Phi_{\overline{y}_1} = \Phi_{\overline{x}_1} r_{\overline{x}_1}^{\overline{y}_1}, \tag{8.9}$$

$$\delta_{\overline{x}_1}^{\widetilde{z}_0} \Phi_{\widetilde{z}_0} = \Phi_{\overline{x}_1} \delta_{\overline{x}_1}^{\widetilde{z}_0} \tag{8.10}$$

$$i_{x_1}^{\overline{x}_1} \Phi_{\overline{x}_1} = \Phi_{x_1} i_{x_1}^{\overline{x}_1} \tag{8.11}$$

hold for all y with $x \to y$ and all z with $z \to x$. For closed points $x \in X$, we set

$$\Phi_{\overline{x}_1} = i^{\overline{x}_1}_{x_1} \varphi_{x_1} \left(i^{\overline{x}_1}_{x_1} \right)^{-1}.$$

Here we have used that, by exactness of (8.6), $i^{\overline{x}_1}_{x_1}$ is invertible as there is no y with $x \to y$. While the condition (8.9) is empty, (8.10) is guarantied by φ being an $\mathcal{R}$-module homomorphism, and (8.11) holds by construction.

Now fix an element $w \in X$ and assume that $\Phi_{\overline{x}_1}$ is defined for all $x < w$ in a way such that (8.9) and (8.10) hold. Using the exact sequence (8.6) and the freeness of $\bigoplus_{w \to x} M(\overline{w}_1)$, we can choose a free subgroup $V \subseteq M(\overline{w}_1)$ such that $M(\overline{w}_1)$ decomposes as an inner direct sum

$$M(\overline{w}_1) = V \oplus M(w_1) \cdot i^{\overline{w}_1}_{w_1}.$$

We will define $\Phi_{\overline{w}_1}$ by specifying the two restrictions $\Phi_{\overline{w}_1}|_V$ and $\Phi_{\overline{w}_1}|_{M(w_1) \cdot i^{\overline{w}_1}_{w_1}}$. Consider the diagram

(8.12)
$$\begin{tikzcd}
V \arrow[r, tail] \arrow[dr, dotted] & M(\overline{x}_1) \arrow[r, "{\left(r^{\overline{y}_1}_{\overline{x}_1},\, -\delta^{\widetilde{z}_0}_{\overline{x}_1}\right)}"] & \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \arrow[r, "{\begin{pmatrix} \delta^{\widetilde{x}_0}_{\overline{y}_1} \\ i^{\widetilde{x}_0}_{\widetilde{z}_0} \end{pmatrix}}"] \arrow[d, "{\left((\Phi_{\overline{y}_1}), (\Phi_{\widetilde{z}_0}) \right)}"] & M(\widetilde{x}_0) \arrow[d, "\Phi_{\widetilde{x}_0}"] \\
& N(\overline{x}_1) \arrow[r, "{\left(r^{\overline{y}_1}_{\overline{x}_1},\, -\delta^{\widetilde{z}_0}_{\overline{x}_1}\right)}"'] & \bigoplus_{x \to y} N(\overline{y}_1) \oplus \bigoplus_{z \to x} N(\widetilde{z}_0) \arrow[r, "{\begin{pmatrix} \delta^{\widetilde{x}_0}_{\overline{y}_1} \\ i^{\widetilde{x}_0}_{\widetilde{z}_0} \end{pmatrix}}"'] & N(\widetilde{x}_0)
\end{tikzcd}$$

By assumption, the rows of this diagram are exact and the right-hand square commutes. We can therefore choose a homomorphism $\Phi_{\overline{x}_1}|_V \colon V \to N(\overline{x}_1)$ such that the left-hand pentagon commutes.

By exactness of (8.6), $i^{\overline{x}_1}_{x_1}$ is injective. Its corestriction onto its image $M(x_1) \cdot i^{\overline{x}_1}_{x_1}$ is thus an isomorphism. We may therefore define the restriction $\Phi_{\overline{x}_1}|_{M(x_1) \cdot i^{\overline{x}_1}_{x_1}}$ in the unique way which makes the following diagram commute:

(8.13)
$$\begin{tikzcd}
M(x_1) \arrow[r, "i^{\overline{x}_1}_{x_1}"] \arrow[d, "\varphi_{x_1}"] & M(x_1) \cdot i^{\overline{x}_1}_{x_1} \arrow[d, "\Phi_{\overline{x}_1}|_{M(x_1) \cdot i^{\overline{x}_1}_{x_1}}"] \\
N(x_1) \arrow[r, "i^{\overline{x}_1}_{x_1}"] & N(x_1) \cdot i^{\overline{x}_1}_{x_1}
\end{tikzcd}$$

We have to check that $\Phi_{\overline{w}_1} = (\Phi_{\overline{w}_1}|_V, \Phi_{\overline{w}_1}|_{M(w_1) \cdot i^{\overline{w}_1}_{w_1}})$ fulfills (8.9) and (8.10) (with x replaced with w). This is true on V because of the commutativity of the left-hand side of (8.12). It is also true on the second summand: by (8.3), both sides of (8.9) vanish on this subgroup; (8.10) follows again from φ being an $\mathcal{R}$-module homomorphism; and (8.11) holds by construction. This completes the induction step.

The claim, that Φ is an isomorphism whenever φ is, follows from a repeated application of the five-lemma.

Finally, our statement on positivity is obvious because only the K_0-groups carry an order and, by our definition, $\Phi_{\widetilde{x}_0} = \varphi_{\widetilde{x}_0}$.

□

Corollary 8.14. *Assume that X is a BDP space. Let M and N be exact, real-rank-zero-like $\mathcal{ST}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists an $\mathcal{ST}$-module homomorphism $\Phi\colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$, and if φ is an isomorphism then so is Φ.*

If $M = \mathrm{FK}_{\mathcal{ST}}(A)$ and $N = \mathrm{FK}_{\mathcal{ST}}(B)$ for C^*-algebras A and B over X with real rank zero, then Φ is an order-isomorphism if and only if φ is.

Remark 8.15. The non-UPP space $\mathcal{D} = \{1, 2, 3, 4\}$ defined by $4 \to 3, 4 \to 2, 3 \to 1, 2 \to 1$ should be mentioned here since it is the only known example of a finite T_0-space X for which there exist *real rank zero* Kirchberg X-algebras with simple subquotients in the bootstrap class that are not $\mathrm{KK}(X)$-equivalent but have isomorphic filtered K-theory, cf. [1, 2].

It turns out that if one adds to the assumptions that the K_1-groups are free, then for such C^*-algebras over $\mathcal{D}$, isomorphisms on the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ lift to $\mathrm{KK}(\mathcal{D})$-equivalences and thereby to $\mathcal{D}$-equivariant *-isomorphisms, by the classification result of Eberhard Kirchberg.

In [2], the second-named author constructed a refinement FK' of filtered K-theory over $\mathcal{D}$ and showed that, for nuclear, separable C^*-algebras over $\mathcal{D}$ with simple subquotients in the bootstrap class, isomorphims on FK' lift to $\mathrm{KK}(\mathcal{D})$-equivalences. Using the same techniques as in the proof of Theorem 8.8, one can show that for real rank zero C^*-algebras A and B over $\mathcal{D}$ with $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ free for all $x \in \mathcal{D}$, any (positive) isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ can be (non-uniquely) extended to a (positive) isomorphism $\mathrm{FK}'(A) \to \mathrm{FK}'(B)$.

For such C^*-algebras, the refined filtered K-theory FK' consists of the groups and maps in the following diagram, where Y_i denotes the group $\mathrm{FK}_Y^i(A)$,

$$\begin{tikzcd}
 & 24_1 \arrow[dr] & & 2_1 \arrow[dr] & & 12_1 \arrow[dr] & & 34_0 \arrow[dr] & & 3_0 \arrow[dr] & & 13_0 \arrow[dr] & \\
4_1 \arrow[ur] \arrow[dr] & & 234_1 \arrow[ur] \arrow[dr] \arrow[r, tail] & 1234_1 \arrow[r, tail] & 123_1 \arrow[ur] \arrow[dr] \arrow[r, two heads] & 4_0 \arrow[r, two heads] & 1 \setminus 4_1 \arrow[ur] \arrow[dr] \arrow[r, two heads] & 1_1 \arrow[r, two heads] & 234_0 \arrow[ur] \arrow[dr] \arrow[r, tail] & 1234_0 \arrow[r, tail] & 123_0 \arrow[ur] \arrow[dr] & & 1_0 \\
 & 34_1 \arrow[ur] & & 3_1 \arrow[ur] & & 13_1 \arrow[ur] & & 24_0 \arrow[ur] & & 2_0 \arrow[ur] & & 12_0 \arrow[ur] &
\end{tikzcd}$$

together with the group $1 \backslash 4_0$ that turns out to be naturally isomorphic to the direct sum of 4_1 and 1_0. The reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ consists of the sequences $3_1 \to 4_0 \to 34_0$, $2_1 \to 4_0 \to 24_0$, $1_1 \to 234_0 \to 1234_0$ together with the maps $34_0 \to 234_0$ and $24_0 \to 234_0$ and the group 4_1.

For each part of the diagram of the form

$$\begin{tikzcd}
 & Z_1 \arrow[dr] & \\
Y \arrow[ur] \arrow[r] \arrow[dr] & Z_2 \arrow[r] & W \\
 & Z_3 \arrow[ur] &
\end{tikzcd}$$

the sequence $Y \to Z_1 \oplus Z_2 \oplus Z_3 \to W$ is exact. Using this, isomorphisms on the remaining K_0-groups in FK' can therefore be constructed as the maps induced on cokernels (and for $1 \setminus 4_0$ on the direct sum of 4_1 and 1_0), and isomorphisms on the remaining K_1-groups can be constructed by choosing split-maps since the relevant groups are free, by the same techniques as in the proof of Theorem 8.8.

The construction should be carried out from right to left, beginning with $1 \setminus 4_1$ and ending with 24_1 and 34_1.

9. Range of reduced filtered K-theory

Let X be an arbitrary, finite T_0-space. Let E be a countable graph and assume that all vertices in E are regular and support at least two cycles. Recall that a cycle is an edge whose source equals its range. Recall also that the saturated, hereditary subsets of E^0 correspond to ideals in $C^*(E)$. Then all subsets of E^0 are saturated, hence a continuous map $\mathrm{Prim}(C^*(E)) \to X$ corresponds to a map $\psi\colon E^0 \to X$ satisfying $\psi(s(e)) \geq \psi(r(e))$ for all $e \in E^1$.

Assume that such a ψ is given, i.e., that $C^*(E)$ is a C^*-algebra over X. Then $\mathrm{FK}_{\mathcal{R}}(C^*(E))$ can be computed in the following way. Define for each $F \subseteq X$ a matrix $D_F \in M_{\psi^{-1}(F)}(\mathbb{Z}_+)$ as $D_F = A_F - 1$ where A_F is defined as

$$A_F(v,w) = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|.$$

Let $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$ be given, and define $C = Y \setminus U$. Then by [7], the six-term exact sequence induced by $C^*(E)(U) \hookrightarrow C^*(E)(Y) \twoheadrightarrow C^*(E)(C)$ is naturally isomorphic to the sequence

$$\begin{tikzcd}
\operatorname{coker} D_U \arrow[r] & \operatorname{coker} D_Y \arrow[r] & \operatorname{coker} D_C \arrow[d, "0"] \\
\ker D_C \arrow[u, "D_Y|^{\psi^{-1}(U)}_{\psi^{-1}(C)}"] & \ker D_Y \arrow[l] & \ker D_U \arrow[l]
\end{tikzcd}$$

induced, via the Snake Lemma, by the commuting diagram

$$\begin{tikzcd}
\mathbb{Z}^{\psi^{-1}(U)} \arrow[r] \arrow[d, "D_U"] & \mathbb{Z}^{\psi^{-1}(Y)} \arrow[r] \arrow[d, "D_Y"] & \mathbb{Z}^{\psi^{-1}(C)} \arrow[d, "D_C"] \\
\mathbb{Z}^{\psi^{-1}(U)} \arrow[r] & \mathbb{Z}^{\psi^{-1}(Y)} \arrow[r] & \mathbb{Z}^{\psi^{-1}(C)}.
\end{tikzcd}$$

Given a map $\psi\colon E^0 \to X$, one can define matrices D_F as above. Then $C^*(E)$ is a C^*-algebra over X, via ψ, if and only if $D_X|^{\psi^{-1}(z)}_{\psi^{-1}(y)}$ vanishes when $y \not< z$. And if furthermore $D_X|^{\psi^{-1}(z)}_{\psi^{-1}(y)}$ is non-zero whenever $y < z$, then $C^*(E)$ is tight over X.

The following theorem by Søren Eilers, Mark Tomforde, James West and the third named author, determines the range of filtered K-theory over the two-point space $\{1,2\}$ with $2 \to 1$. We quote it here to apply it in the proof of Theorem 9.2.

Theorem 9.1 ([8, 4.3 & 4.7])**.** *Let $\mathcal{E}$*

$$\begin{tikzcd}
G_1 \arrow[r, "\varepsilon"] & G_2 \arrow[r, "\gamma"] & G_3 \arrow[d, "0"] \\
F_3 \arrow[u, "\delta"] & F_2 \arrow[l, "\gamma'"'] & F_1 \arrow[l, "\varepsilon'"']
\end{tikzcd}$$

be an exact sequence of abelian groups with F_1, F_2, F_3 free. Suppose that there exists row-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{1, 2, \ldots, \infty\}$ with isomorphisms

$$\alpha_1\colon \operatorname{coker} A \to G_1, \quad \beta_1\colon \ker A \to F_1,$$

$$\alpha_3\colon \operatorname{coker} B \to G_3, \quad \beta_3\colon \ker B \to F_3.$$

Then there exists a row-finite matrix $Y \in M_{n_3,n_1'}(\mathbb{Z})$ *and isomorphisms*

$$\alpha_2\colon \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to G_2, \quad \beta_2\colon \ker\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to F_2$$

such that $(\alpha_1, \alpha_2, \alpha_3\beta_1, \beta_2, \beta_3)$ *gives an isomorphism of complexes from the exact sequence*

$$\begin{tikzcd}
\operatorname{coker} A \arrow[r, "I"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[r, "P"] & \operatorname{coker} B \arrow[d, "0"] \\
\operatorname{coker} B \arrow[u, "{[Y]}"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[l, "P'"'] & \operatorname{coker} A \arrow[l, "I'"']
\end{tikzcd}$$

where the maps I, I' *and* P, P' *are induced by the obvious inclusions or projections, to the exact sequence* $\mathcal{E}$.

If there exist an $A' \in M_{n_1',n_1}$ *such that* $A'A - 1 \in M_{n_1',n_1'}(\mathbb{Z}_+)$, *then* Y *can be chosen such that* $Y \in M_{n_3,n_1'}(\mathbb{Z}_+)$. *If furthermore a row-finite matrix* $Z \in M_{n_3,n_1'}(\mathbb{Z})$ *is given, then* Y *can be chosen such that* $Y - Z \in M_{n_3,n_1'}(\mathbb{Z}_+)$.

As subquotients of graph algebras are graph algebras, the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ of a graph algebra A over X will satisfy that the group $\mathrm{K}_1(A(x))$ is free for all $x \in X$. Combining this with the following theorem, we get at complete description of the range of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$.

Theorem 9.2. *Let* M *be an exact* $\mathcal{R}$*-module with* $M(x_1)$ *free for all* $x \in X$. *Then there exists a countable graph* E *satisfying that all vertices in* E *are regular and support at least two cycles, and that* $C^*(E)$ *is tight over* X *and has* $\mathrm{FK}_{\mathcal{R}}(C^*(E))$ *isomorphic to* M.

The graph E *can be chosen to be finite if (and only if)* $M(x_1)$ *and* $M(\widetilde{x}_0)$ *are finitely generated, and the rank of* $M(x_1)$ *coincides with the rank of the cokernel of* $i\colon M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)$, *for all* $x \in X$.

Proof. For each $x \in X$, choose by [8, 3.3] a matrix $D_x \in M_{V_x}(\mathbb{Z}_+)$, where V_x is a countable, non-empty set, satisfying that $\ker D_x$ is isomorphic to $M(x_1)$ and $\operatorname{coker} D_x$ is isomorphic to $M(x_0) = \operatorname{coker}(M(\widetilde{\partial} x_0) \xrightarrow{i} M(\widetilde{x}_0))$, and that all vertices in the graph E_{D_x+1} are regular and support at least two cycles. If $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)$, then the set V_x can be chosen to be finite. Let $\varphi_{x_1}\colon M(x_1) \to \ker D_x$ and $\varphi_{x_0}\colon M(x_0) \to \operatorname{coker} D_x$ denote the isomorphisms.

For each $y, z \in X$ with $y \neq z$ we desire to construct a matrix $H_{yz}\colon \mathbb{Z}^{V_z} \to \mathbb{Z}^{V_y}$ with non-negative entries satisfying that H_{yz} is non-zero if and only if $y > z$, and satisfying that for each $x \in X$ there exists isomophism $\varphi_{\widetilde{\partial} x_0}$ and $\varphi_{\widetilde{x}_0}$ making the diagrams

$$\begin{tikzcd}
M(\widetilde{\partial}x_0) \arrow[rr, "i"] \arrow[dr, "\varphi_{\widetilde{\partial}x_0}"] & & M(\widetilde{x}_0) \arrow[rr] \arrow[d, "\varphi_{\widetilde{x}_0}"] & & M(x_0) \arrow[dl, "\varphi_{x_0}"] \\
& \operatorname{coker} D_{\widetilde{\partial}(x)} \arrow[r] & \operatorname{coker} D_{\widetilde{\{x\}}} \arrow[r] & \operatorname{coker} D_x \arrow[d, "0"] & \\
& \ker D_x \arrow[u, "D_{\widetilde{\{x\}}}|^{\varphi^{-1}(\widetilde{\partial}(x))}_{\varphi^{-1}(x)}"] & \ker D_{\widetilde{\{x\}}} \arrow[l] & \ker D_{\widetilde{\partial}(x)} \arrow[l] & \\
M(x_1) \arrow[uuu, "\delta"] \arrow[ur, "\varphi_{x_1}"] & & & &
\end{tikzcd} \tag{9.3}$$

and

$$\begin{tikzcd}
M(\widetilde{y}_0) \arrow[rr, "i"] \arrow[dr, "\varphi_{\widetilde{y}_0}"] & & M(\widetilde{\partial}x_0) \arrow[d, "\varphi_{\widetilde{\partial}x_0}"] & \\
& \operatorname{coker} D_{\widetilde{\{y\}}} \arrow[r] & \operatorname{coker} D_{\widetilde{\partial}(x)} \arrow[r] & \operatorname{coker} D_{\widetilde{\partial}(x)\setminus\widetilde{\{y\}}} \arrow[d, "0"] \\
& \ker D_{\widetilde{\partial}(x)\setminus\widetilde{\{y\}}} \arrow[u, "D_{\widetilde{\partial}(x)}|^{\varphi^{-1}(\widetilde{\{y\}})}_{\varphi^{-1}(\widetilde{\partial}(x)\setminus\widetilde{\{y\}})}"] & \ker D_{\widetilde{\partial}(x)} \arrow[l] & \ker D_{\widetilde{\{y\}}} \arrow[l]
\end{tikzcd} \tag{9.4}$$

commute when $y \to x$, and where $D_F \in M_{V_F}(\mathbb{Z}_+)$ for each $F \subseteq X$ is defined as

$$D_F(v,w) = \begin{cases} D_x(v,w) & v,w \in V_x \\ H_{yz}(v,w) & v \in V_y, w \in V_x, x \neq y \end{cases}$$

where $V_F = \bigcup_{y\in F} V_y$. The constructed graph E_{D_X+1} will then have the desired properties.

Let $U \in \mathbb{O}(X)$ and assume that for all $z, y \in U$, the matrices H_{yz} and isomorphisms $\varphi_{\widetilde{\partial}y_0}$ and $\varphi_{\widetilde{y}_0}$ have been defined and satisfy that the diagrams (9.3) and (9.4) commute for all $x, y \in U$ with $y \to x$. Let x be an open point in $X \setminus U$ and let us construct isomorphisms $\varphi_{\widetilde{\partial}x_0}$ and $\varphi_{\widetilde{x}_0}$, and for all $y \in \widetilde{\partial}(x)$ non-zero matrices H_{yx}, making the diagrams (9.3) and (9.4) commute.

Consider the commuting diagram

$$\begin{tikzcd}
\bigoplus M(\widetilde{z}_0) \arrow[r] \arrow[d, "(\varphi_{\widetilde{z}_0})"] & \bigoplus_{y\to x} M(\widetilde{y}_0) \arrow[r] \arrow[d, "(\varphi_{\widetilde{y}_0})"] & M(\widetilde{\partial}x_0) \arrow[r] \arrow[d, dotted] & 0 \\
\bigoplus \operatorname{coker} D_{\widetilde{\{z\}}} \arrow[r] & \bigoplus_{y\to x} \operatorname{coker} D_{\widetilde{\{y\}}} \arrow[r] & \operatorname{coker} D_{\widetilde{\partial}(x)} \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by exactness of M, and the bottom row is exact by exactness of $\mathrm{FK}(C^*(E_{D_{\widetilde{\partial}(x)}+1}))$. An isomorphism $\varphi_{\widetilde{\partial}x_0}\colon M(\widetilde{\partial}x_0) \to \operatorname{coker} D_{\widetilde{\partial}(x)}$ is therefore induced. By construction, (9.4) commutes for all $y \to x$.

Now consider the commuting diagram

$$\begin{tikzcd}
M(\widetilde{\partial}x_0) \arrow[rr, "i"] \arrow[dr, "\varphi_{\widetilde{\partial}x_0}"] & & M(\widetilde{x}_0) \arrow[rr] \arrow[d, equal] & & M(x_0) \arrow[dl, "\varphi_{x_0}"'] \\
& \operatorname{coker} D_{\widetilde{\partial}(x)} \arrow[r] & M(\widetilde{x}_0) \arrow[r] & \operatorname{coker} D_x \arrow[d, "0"] & \\
& \ker D_x \arrow[u] & F \arrow[l] & \ker D_{\widetilde{\partial}(x)} \arrow[l] & \\
M(x_1) \arrow[uuu, "\delta"] \arrow[ur, "\varphi_{x_1}"] & & & &
\end{tikzcd}$$

where a free group F and maps in to and out of it have been choosen so that the inner six-term sequence is exact. Apply [8] to the inner six-term exact sequence to get non-zero matrices H_{yx} for all $y \in \widetilde{\partial}(x)$ realising the sequence, i.e., making (9.3) commute. □

Corollary 9.5. *Let X be a finite T_0-space and assume that $\mathrm{FK}_{\mathcal{R}}$ is a complete invariant for purely infinite, separable, nuclear, real rank zero C^*-algebras that are tight over X and satisfy that for all $x \in X$, $A(x)$ is in the bootstrap class and $\mathrm{K}_1(A(x))$ is free.*

Let $I \hookrightarrow A \twoheadrightarrow B$ be an extension of C^*-algebras where I and B are stably isomorphic to Cuntz-Krieger algebras, A has primitive ideal space X, and the induced map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes. Then A is stably isomorphic to a Cuntz-Krieger algebra.

Proof. As I and B have real rank zero and the boundary map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes, A has real rank zero. As for each $x \in X$, $A(x)$ is a simple subquotient of either I or B, $\mathrm{K}_1(A(x))$ is free and $\operatorname{rank} \mathrm{K}_0(A(x)) = \operatorname{rank} \mathrm{K}_1(A(x)) < \infty$. Hence there exists a Cuntz-Krieger algebra D satisfying $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(D)$. □

10. Main result

Combining our results with the completeness of filtered K-theory over accordion spaces, we get the following characterization of purely infinite graph algebras, and of Cuntz-Krieger algebras.

Theorem 10.1. *Let X be an accordion space. There are bijections between the following sets:*

- *stable isomorphism classes of tight, purely infinite graph algebras over X,*
- *isomorphism classes of Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class, and satisfying that $\mathrm{K}_1\big(A(\{x\})\big)$ is free for all $x \in X$,*
- *isomorphism classes of countable, exact, real-rank-zero-like $\mathcal{NT}$-modules M with $M(\{x\},1)$ free for all $x \in X$,*
- *isomorphism classes of countable, exact, real-rank-zero-like, $\mathcal{ST}$-modules M with $M(\{x\},1)$ free for all $x \in X$,*
- *isomorphism classes of countable, exact $\mathcal{B}$-modules M with $M(x_1)$ free for all $x \in X$,*

- *isomorphism classes of countable, exact $\mathcal{R}$-modules M with $M(\overline{x}_1)$ free for all $x \in X$.*

Corollary 10.2. *Let X be an accordion space. There are bijections between the following sets:*

- *isomorphism classes of tight Cuntz Krieger algebras over X,*
- *isomorphism classes of Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class, and with finitely generated filtered* K*-theory such that* $\mathrm{K}_1\big(A(\{x\})\big)$ *is free for all $x \in X$ and* $\operatorname{rank}\mathrm{K}_0\big(A(Y)\big) = \operatorname{rank}\mathrm{K}_1\big(A(Y)\big) < \infty$ *for every $Y \in \mathbb{LC}(X)$,*
- *isomorphism classes of countable, exact, real-rank-zero-like $\mathcal{NT}$-modules M with $M(\{x\},1)$ free for all $x \in X$ and*
$$\operatorname{rank}(M(\{x\},0)) = \operatorname{rank}(M(\{x\},1)) < \infty,$$
- *isomorphism classes of countable, exact, real-rank-zero-like, $\mathcal{ST}$-modules M with $M(\{x\},1)$ free for all $x \in X$ and*
$$\operatorname{rank}(M(\{x\},0)) = \operatorname{rank}(M(\{x\},1)) < \infty,$$
- *isomorphism classes of countable, exact $\mathcal{B}$-modules M with $M(x_1)$ free for all $x \in X$ and*
$$\operatorname{rank}(\operatorname{coker}(\bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{x}_0))) = \operatorname{rank}(x_1) < \infty,$$
- *isomorphism classes of countable, exact $\mathcal{R}$-modules M with $M(\overline{x}_1)$ free for all $x \in X$ and*
$$\operatorname{rank}(\operatorname{coker}(M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0))) = \operatorname{rank}(M(x_1)) < \infty.$$

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